A Mathematical Model of Political Affiliation

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Department of Mathematics Technical Report

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Abstract

This work explores voting trends by analyzing how individuals form their political affiliations during a presidential campaign. Using a variation of the traditional epidemiological model, we construct an ODE model that represents the transition of potential voters through various levels of political interest in either the Republican or Democratic Party. We analyze variations of our model to understand the impact of various interactions between potential voters, such as those between politically-charged and apathetic individuals during a presidential campaign. Finally, we calculate and interpret threshold values to determine the stability of the steady state solutions.

1 Introduction

For more than 100 years, the Democrats and the Republicans have dominated the American Political System, winning every Presidential election since 1853. Yet, neither party has ever had the ability to largely outnumber the other in membership levels nor has one lone party ever successfully retained the position of Commander-in-Chief over a considerable amount of time. It has been these attributes about the two parties that have driven many political scientists to study the components that lead to an individual's support of a candidate. With this in mind, we create a mathematical model to study the factors that lead to the formation of various political affiliations and evaluate the trends in voting patterns, especially during Presidential campaigns. The purpose of our research is to explore these voting trends by analyzing how individuals form their political affiliations. We will use ordinary differential equations to model the influences that causes individuals to transition through voting groups. Given that a majority of the voters in the US over the last four to five presidential elections either vote Democrat or Republican in a presidential campaign, we consider the situation when we have two political parties [15]. We will examine various cases in our model to understand the impact of various interactions such as those between politically-charged and apathetic individuals during a presidential election.

1.1 Types of Political Affiliations

When analyzing the political system, it is important to identify and quantify the individuals who take part in the campaigning process. Since we are considering a presidential election, only the voting-age population (VAP) is relevant to our analysis and model. As of 2004, there were an estimated 142,005,000 million citizens in the VAP [6]. Our analysis will assume that all individuals in this category are eligible to vote and therefore make up the entire population in our mathematical system.

In order to mathematically model voting trends, we make a few key assumptions. As mentioned earlier, we consider only two political parties and consider the influence of the remaining voters as negligible. While this assumption is mathematically based on the proportions of people in the respective political parties, we understand the potentially significant effects a third party can have, as evidenced in the 2000 presidential election [15]. However, we believe understanding the dynamics of the two-party system is crucial to an overall understanding of the complete political situation.

Before we can mathematically model political affiliations, we need to define which affiliations are possible. Consider an individual who forms an opinion about Political Party A, assuming that the only alternative group in the system is Party B. We assume that the individual can take one of three stances: they can either support the group, oppose the group i.e.,(support Party B), or remain Undecided. In the undecided category, we also include potential voters who could be considered apathetic or supporters of a third party. We further assume that there are two levels of support that an individual can exhibit towards a political party; they can either mildly support the party or strongly support it. These two levels can also be exhibited when an individual opposes a political party. With this in mind, we have five states that an individual may enter when forming opinions about two parties in any political system:

1) Undecided/Apathetic/Other
2) Mildly Supportive of Party A
3) Strongly Supportive of Party A
4) Mildly Supportive of Party B
5) Strongly Supportive of Party B.

These stances form the foundation for the classification of our states in the American Political System. The two parties, the Democrats and the Republicans, provide the general options for which an individual in the system can choose. In our case, we define undecided/Apathetic/Other as the Susceptible Class, denoted by S. Individuals in this category are either undecided about which candidate they favor, apathetic toward politics, or have chosen to support a third party candidate. Individuals classified as Mildly Supportive of Party A in the previous example are classified as Moderate Democrats denoted by E_D in our model. Individuals belong to this category if they are not actively promoting the Democratic candidate, but plan to vote for the Democratic candidate on election day. When we say an individual is not "actively" supporting his or her party's candidate we mean that the person is not campaigning, advertising, participating in telethons, putting up posters, or initiating political discussion about the presidential election with other individuals in any way. However, a person who shares his or her political views only when asked about those views is included in the Moderate Category if that person is not otherwise actively supporting a political candidate. Those who would be classified as Mildly Supportive of Party B in the previous example are classified as Moderate Republicans denoted by E_R in our model. Moderate Republicans are classified in the same way as Moderate Democrats except that these individuals plan to vote for the Republican presidential candidate. Individuals who would be included in the Strongly Supportive of Party A category in the previous general example, are characterized as Fanatical Democrats denoted by F_D in our model. A person is considered a Fanatical Democrat if that person plans to vote for the Democratic presidential candidate and actively supports that candidate. Once again by "active" support we mean trying to convince others to vote Democrat via advertisements, telethons, rallies, instigating political debate, or other methods. Fanatical Republicans, denoted by F_R are classified in the same way as Fanatical Democrats except the individuals in the Fanatical Republican plan to vote for the Republican presidential candidate. These voting classes are summarized below:

1) Susceptible - S (Undecided/Apathetic/Other)
2) Moderate Democratic - E_D (Mildly Supportive of Democrats)
3) Fanatical Democrat - F_D (Strongly and Actively Supportive of Democrats)
4) Moderate Republican - E_R (Mildly Supportive of Republicans)
5) Fanatical Republican - F_R (Strongly and Actively Supportive of Republicans)

1.2 Factors that Cause Individuals to Form and Change Political Affiliations

During the time period between when the presidential candidates have won their respective primary elections and the November Presidential election, the potential voters move between groups that we identify as the Susceptible Individuals, (S), Moderate Democratic/Republican Voters, (E_D/E_R), or Fanatical Democrats/ Republicans, (F_D/F_R). Though the time period considered is over the course of approximately two months, there is sufficient time for the potential voters in our system to be influenced to change their political stances. According to Holbrook, there has been much evidence suggesting that there is a significant change in public opinion during the campaign season and that the change in public opinion can be due to campaign events during the time [9]. Thus, the potential voters can be influenced to become more active or discouraged in a way that lessens the support the voter had initially given the candidate. The factors that influence a person's decision to support or not to support a candidate are based on the type of interactions one has with individuals in the other classes and the individuals' background and personal convictions.

One of the contributing factors in an individual's decision to change his/her voting class is one's personal influence. This includes, for example, individuals' religious values, their socio-economic status, their family upbringing, and their cultural values. Personal influence also includes the incumbency factor since whether a candidate has already been a president affects how a person views that candidate [17]. This self-motivation factor also takes into consideration an individual's ideology, values, and behavior. When a person makes a decision to consider voting for a candidate, or when an individual decides to become more supportive of a candidate (free of any outside influence or interaction with any of the existing groups), we say that this person moved from one group to another based on their personal judgements about an issue. We assume these views have been made based on the individual's pre-existing ideas and opinions. Additionally, there is another type of personal influence that affects individuals' classifications in our system – their obligations. For example, if people are very busy at work or at home then they may choose to become less active in politics or not go to the voting station on election day because they are too busy trying to meet all their personal obligations. In our model, these personal factors are identified by the p_i terms. Just to reiterate, this factor does not take into consideration the interactions an individual may have with someone from another group that may influence their political ideas.

External influence, which comes about by interaction between individuals from two different groups, can happen in different forms. In our model, we define external influence as telecommunication, which includes television, e-mail, telephone conversations, radio, or any other method through which information about the presidential candidate is spread and as person to person interaction. We assume that these external influences affect an individual's political beliefs in two ways. Individuals in a group can be influenced either positively or negatively by people from another group when they interact. We define a positive interaction as when an individual encourages a person from a less active group or an indifferent group to move to a more active group. An example of this is when a Fanatical Democrat interacts with a Moderate Democrat and successfully convinces the Moderate Democrat to become a Fanatic as well. This interaction caused the the once-Moderate Democrat to become more active and supportive of their candidate. We similarly define negative influence as any type of interaction between members of two groups in which members of one group cause a member of the other group to lose interest in their party's candidate. For instance, if a person from the F_R group turns-off an individual from the E_R group, then the individual from the E_R group will go back to being susceptible or will join the Democratic Party and thus will no longer support the Republican candidate.

In summary, our model considers how people move between groups based on interactions with other individuals and personal influence. What influences a person to change a political stance or to become more supportive or less supportive of a candidate depends upon these interactions and personal influences. We take all these considerations and assumptions into account and create a mathematical model that gives some insight into the formation and development of political affiliations in the two month period before a presidential election in the US.

2 Formation of Our Mathematical Model

To formulate our model, it is important to consider the possible interactions that can occur between individuals, the rate at which these interactions occur as well as the effect that the interaction has on the individuals involved. As stated previously in our mathematical model we denote our classes, as follows: S: Susceptible Individuals, E_D: Moderate Democrats, E_R: Moderate Republicans, F_D: Fanatical Democrats, and F_R: Fanatical Republicans. We use traditional epidemiological models such as, [3] and [4] as a framework for our mathematical model and analysis of political affiliation.

We assume there are no births and deaths in the system. This is reasonable since the net change in the VAP over the two month time period before the presidential election is small. In fact, it is negligible with respect to the total VAP. We further assume that in each class all individuals are the same i.e., there is homogeneity within each class. Also, for every interaction, the probability of interacting with a member of a given class is proportional to the number of individuals in that class, relative to the total population. Thus, interactions are modeled via the mass-action law. As an example of a typical interaction given in our flow diagram in Figure 1, consider the rate at which individuals in the S class interact with individuals in the E_R class and then join the E_R class. The maximum amount of interactions that can occur between these individuals can be represented by the product, SE_R. However, it is unreasonable to assume that all individuals in one state interact with all individuals in another state.

We introduce a parameter b_{14} to represent the rate at which these classes interact as well as the probability that the individual leaves the S class and joins the E_R class due to the interaction. We divide by the total population N to represent the proportion of movement out of S due to this interaction. Thus, we obtain our basic expression $(b_{14}\frac{SE_R}{N})$. This expression represents the rate at which movement from S to E_R takes place. However, this only represents the contribution that an SE_R interaction has on moving S to E_R. We must also consider all other interactions that are responsible for the movement of S individuals into the E_R state. (This includes SE_R,SF_D,SF_R). The derivation of the contributions of each of these interactions is analogous to the derivation of the interaction between S and E_R. Since it is not necessarily true that all interactions have the same movement rate, the parameters for each interaction should be different. We denote these parameters as b_i, c_i, d_i, and g_i for all i. The next expression that must be considered is personal influence which is denoted by p_i. This represents the contribution that an individual's self-motivation and beliefs have on identifying with a political affiliation. For example, suppose an individual in the F_R state has a personal influence term, p_i, that a ects their political stances. Then, we will denote the portion that is determined by this individual's personal influence as pF_R. These two expressions form the basis of our system of ordinary differential equations.

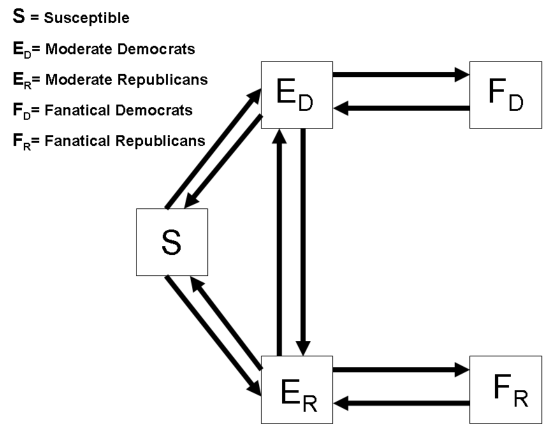


Figure 1

Using this idea to build expressions relating to the transition between classes, we construct a mathematical ideological model representing the formations and development of political affiliation. This ideological model is governed by the following differential equations representing the effects of interactions among eligible voters in a presidential election.

$$
\begin{aligned}
\frac{dS}{dt} &= -\frac{S}{N}(b_1E_R + b_2F_R + b_3E_D + b_4F_D) + p_5E_D + p_4E_R - p_6S - p_3S \\
&+\frac{E_D}{N}(b_6F_D + b_7E_R + b_8F_R) + \frac{E_R}{N}(b_{10}F_R + b_{11}E_D + b_{12}F_D) \\
&-\frac{S}{N}(b_{14}E_R + b_{15}F_R + b_{16}E_D + b_{17}F_D) \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dE_D}{dt} &= \frac{S}{N}(b_1E_R + b_2F_R + b_3E_D + b_4F_D) - p_8E_D + p_9F_D \\
&+\frac{E_R}{N}(c_1E_D + c_2F_D + c_3F_R) - \frac{E_D}{N}(c_5E_R + c_6F_R + c_7F_D) \\
&+\frac{F_D}{N}(c_9E_R + c_{10}F_R) - \frac{E_D}{N}(c_{12}F_D - g_3E_R + g_4F_R) \\
&-\frac{E_D}{N}(b_6F_D + b_7E_R + b_8F_R) - p_5E_D + p_7E_R + p_6S - p_{10}E_D \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dE_R}{dt} &= \frac{S}{N}(b_{14}E_R + b_{15}F_R + b_{16}E_D + b_{17}F_D) + p_3S - p_4E_R - p_7E_R \\
&-\frac{E_R}{N}(d_1F_R + g_1E_D + g_2F_D) - \frac{E_R}{N}(c_1E_D + c_2F_D + c_3F_R) \\
&+\frac{E_D}{N}(c_5E_R + c_6F_R + c_7F_D) + p_8E_D + \frac{F_R}{N}(d_3E_D + d_4F_D) \\
&-\frac{E_R}{N}(b_{10}F_R + b_{11}E_D + b_{12}F_D) - p_{12}E_R + p_{13}F_R \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dF_D}{dt} &= \frac{E_D}{N}(c_{12}F_D + g_3E_R + g_4F_R) - p_9F_D \\
&-\frac{F_D}{N}(c_9E_R + c_{10}F_R) + p_{10}E_D \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dF_R}{dt} &= \frac{E_R}{N}(d_1F_R + g_1E_D + g_2F_D) - p_{13}F_R \\
&-\frac{F_R}{N}(d_3E_D + d_4F_D) + p_{12}E_R \qquad (5)
\end{aligned}
$$

2.1 Normalizing the Governing Equations of Our Model

It is often convenient to consider fractions of a population rather than whole population numbers. We achieve this mathematically by normalizing our equations and renaming the variables to represent fractions of the overall population. The total population is $N = S + E_D + E_R + F_D + F_R$. Summing equations (1)-(5) gives $\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE_D}{dt} + \frac{dE_R}{dt} + \frac{dF_D}{dt} + \frac{dF_R}{dt} = 0$, which shows that our population is constant. Thus, we can choose to ignore one of the variables. We define the quantity $s = \frac{S}{N}$ where s represents the proportion of the total population that belongs in the Susceptible State.

Similarly, we can define $e_d = \dfrac{E_D}{N}$ to be the proportion of population in the Moderate Democrat Class, (E_D), $e_r = \dfrac{E_R}{N}$ to be the proportion of the total population in the Moderate Republican Class, (E_R), $f_d = \dfrac{F_D}{N}$ to be the proportion in the Fanatical Democrat Group, (F_D), and $f_r = \dfrac{F_R}{N}$ to be the proportion in the Fanatical Republican Group, (F_R).

Substituting these normalized variables into our system and simplifying yields our normalized system. For brevity we show only the normalized Susceptible equation,

$$\frac{ds}{dt} = -s(b_1 e_r + b_2 f_r + b_3 e_d + b_4 f_d) - p_6 s + e_d(b_6 f_d + b_7 e_r + b_8 f_r) + p_5 e_d$$
$$+ e_r(b_{10} f_r + b_{11} e_d + b_{12} f_d) + p_4 e_r - s(b_{14} e_r + b_{15} f_r + b_{16} e_d + b_{17} f_d) - p_3 s$$

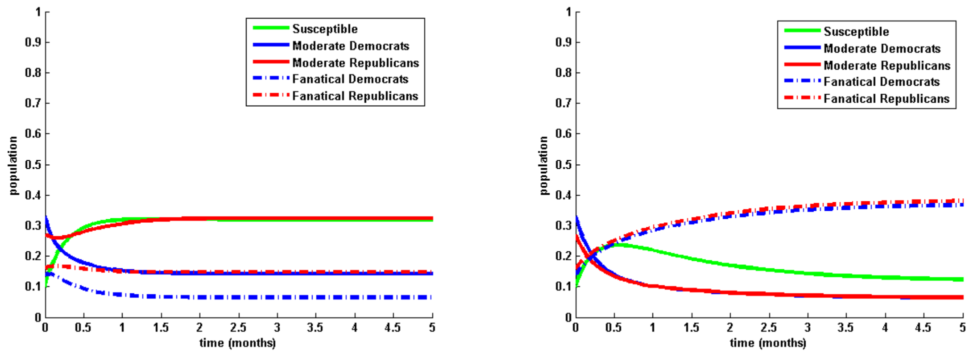
The remaining substitutions have an analogous effect. For convenience, we rename the normalized variables such that $s \to S$, $e_d \to E_D$, $e_r \to E_R$, $f_d \to F_D$, and $f_r \to F_R$.

2.2 Preliminary Analysis of Our Mathematical Model

We begin our analysis by trying to gain a preliminary understanding of the dynamics of our ODE model. As with many other types of nonlinear ODE's, we can consider the long-term behavior of our system. However, unlike the standard epidemiological models, we do not have an IFE, an Idea-Free Equilibrium, in which the population consists only of those in the Susceptible class. This is due to the presence of personal influence terms which represent self-motivation and have no analogy in a typical disease model. Numerical exploration with various parameter values seems to indicate the presence of non-zero stable equilibrium values for each of the classes.

In terms of our original model, we can interpret these equilibria as the proportions of the population that would vote for a given candidate, in which our population is divided into various classes, with each class containing a certain percentage of the total population. Thus if we consider Figure 2a, we begin with 10% of the population as Susceptible Individuals, 33% as Moderate Democrats, 26% as Moderate Republicans, 14% as Fanatical Democrats, and 17% as Fanatical Republicans. After some time, we observe that the proportions have settled down to 33% of the population as Susceptible Individuals, 15% as Moderate Democrats, 15% as Moderate Republicans, 4% as Fanatical Democrats, and 15% as Fanatical Republicans.

If we consider Figure 2b, we have the situation where the final voting populations are identical, a situation that is often found in democratic countries with two main parties. Mathematically, we say the equilibria are symmetric because $E_R = E_D$ and $F_R = F_D$. We will consider this situation where $E_R = E_D$ and $F_R = F_D$ before considering other special cases involving restrictions on various parameters.



(a) General

(b) General Symmetric

Figure 2: Figure 2(a) is a numerical representation of a thoroughly distributed population while Figure 2(b) is a symmetrically distributed population.

3 Symmetric Model

3.1 Origin of Symmetric Movement

In the 2000 presidential election, the Republican candidate George W. Bush won the election by a very small margin. In fact, a report by the Federal Election Committee shows that the popular vote was split in two with approximately 49% of voters choosing Gore, the Democratic candidate, and approximately 48% electing Bush, the Republican candidate [6]. This situation, where the country is evenly divided, is one that we would like to capture in our model. One interpretation of this phenomenon is that, on average, the Democrats and Republicans both influenced individuals in the same way and with the same degree of success.

To mathematically reflect this situation, where a population tends to divide itself almost evenly between Democrats and Republicans, we set analogous parameters equal to each other. For example, in the $\frac{dS}{dt}$ equation, $b_1 S E_R$ corresponds to the interactions between S and E_R states that causes the individuals in S to join the E_R Class. Since these two components have similar outcomes (a Moderate individual turning o a Susceptible, who then joins the opposing party), we want to assume that they occur with the same likelihood and movement rate, i.e., $b_1 = b_{16}$. This procedure of equating analogous parameters was done throughout the entire model Table (3.1) illustrates how the number of parameters is reduced in the symmetric case.

3.1 Table of Reduced Parameters

$b_{17} = b_2$	$b_{16} = b_1$	$b_{15} = b_4$	$b_{14} = b_3$	$b_{12} = b_8$	$b_{11} = b_7$
$b_{10} = b_6$	$c_7 = c_3$	$c_6 = c_2$	$c_5 = c_1$	$d_4 = c_{10}$	$d_3 = c_9$
$d_1 = c_{12}$	$g_4 = g_2$	$g_3 = g_1$	$p_8 = p_7$	$p_5 = p_4$	$p_6 = p_3$
$p_{13} = p_9$	$p_{12} = p_{10}$				

This process of equating analogous parameters causes the net strength of the influences of the Republican party and the Democratic party to be the same. Furthermore, if the initial populations of the Republican Party and the Democratic Party in the Fanatic Classes and the Moderate Classes are the same, then the evolutions of these groups can be described by the same set of equations. Thus, we can describe what is happening in the system using three equations rather than five. Hence, we set $E_D = E_R$ and $F_D = F_R$ in equations (1-5). To further simplify the model by reducing the number of parameters, we made the following parameter substitutions: $\ = b_1+b_3$, $\ = b_2+b_4$, $\ _1 = b_6+b_8, \rho = c_9-c_{12}, \epsilon = p_4+p_{10}, \nu = b_7+g_1$. We see that $\frac{dE_D}{dt}$ and $\frac{dE_R}{dt}$ are still identical as are $\frac{dF_D}{dt}$ and $\frac{dF_R}{dt}$. This confirms that we can reduce the model to a system of three differential equations where the Moderate Populations are the same at any given time as are the Fanatic Populations. Since the overall population remains constant, we could again reduce our system further by replacing S with $1 - E_D - E_R - F_D - F_R$. However, because we would like to keep track of the percent of the population in each group, we will consider the three equations and just enforce the constant population constraint separately.

We arbitrarily choose to consider the Democratic groups as opposed to the Republican groups. Thus, the resulting equations are:

$$\frac{dS}{dt} = -2SE_D \quad - 2SF_D \quad - 2p_3S + 2E_DF_D \ _1 + 2E_D\nu - 2g_1E_D^2 + 2E_D\epsilon - 2p_{10}E_D \tag{6}$$

$$\frac{dE_D}{dt} = SE_D \quad +SF_D \quad +p_3S - E_DF_D \ _1 - E_D^2\nu - E_D\epsilon + F_DE_D\rho + c_{10}F_D^2 + p_9F_D - E_Dg_2F_D \tag{7}$$

$$\frac{dF_D}{dt} = p_{10}E_D + g_1E_D^2 + E_Dg_2F_D - F_DE_D\rho - c_{10}F_D^2 - p_9F_D. \tag{8}$$

The first stage in the analysis of this symmetric case is to find the equilibria of the system. The four equilibria (S^*, E_D^*, F_D^*) that result from these equations are:

$$\begin{aligned} E_{s_1} &: (1,0,0) \quad \text{for } p_3 = 0 \\ E_{s_2} &: \left(0, \frac{1}{2}, 0\right) \quad \text{for } b_7 = g_1 = p_4 = p_{10} = 0 \\ E_{s_3} &: \left(0, 0, \frac{1}{2}\right) \quad \text{for } c_{10} = p_9 = 0, \end{aligned}$$

$E_{s_4} : f(V, A, g_1, \ , \ , p_3, g_2, \nu, \rho, c_10, p_9, \ , \ 1, \epsilon, p_{10})$ without restrictions on parameters, where

$$f(V, A, g_1, \;\;, \;\;, p_3, g_2, \nu, \rho, c_{10}, p_9, \;\;, \;\; 1, \epsilon, p_{10}) = \frac{1}{(g_1(A\beta + V \quad + p_3))}(g_1 A V \quad {}_1 + g_1 A\epsilon - A g_2 V \nu$$
$$+A g_2 V g_1 + A V \rho \nu - A V \rho g_1 - A p_{10} \nu +$$
$$c_{10} V^2 \nu - c_{10} V^2 g_1 + p_9 V \nu - p_9 V g_1), A, V \Big) \Big($$

$V = F_D^* = 1 - S^* - 2E_D^*$ and A is a root of the quadratic equation

$$(g_1 Z^2 + (g_2 V - V\rho + p_{10})Z - c_{10}V^2 - p_9 V).$$

When we solve this quadratic equation, we find that:

$$Z = \frac{1}{2g_1}(-g_2 F_D + F_D\rho - p_{10} \pm (g_2^2 F_D^2 - 2g_2 F_D^2 \rho + 2g_2 F_D p_{10} + F_D^2 \rho^2 - 2F_D \rho p_{10} + \cdots$$
$$+p_{10}^2 + 4c_{10} F_D^2 g_1 + 4p_9 F_D g_1)^{1/2}),$$
$$\frac{1}{2g_1}(-g_2 F_D + F_D\rho - p_{10} - (g_2^2 F_D^2 - 2g_2 F_D^2 \rho + 2g_2 F_D p_{10} + F_D^2 \rho^2 - 2F_D \rho p_{10} + \cdots$$
$$+p_{10}^2 + 4c_{10} F_D^2 g_1 + 4p_9 F_D g_1)^{1/2}) \Big) \Big($$

In the following four subsections, we examine the stability of each of these equilibria, $E_{s1} - E_{s4}$.

3.1.1 Stability Analysis of the Equilibrium Point E_{s_1} : $(1, 0, 0)$

Although $(1, 0, 0)$ is an equilibrium point when $p_3 = 0$, let us first consider the more extreme situation where all the personal influence terms are ignored. The eigenvalues of the Jacobian matrix are:

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 =$$

Because $\;\; > 0$ we can easily conclude that the equilibrium point is unstable. That is, without any personal influence, the system will not remain in the situation where no one votes for the Democratic or Republican candidates as long as there is someone in one of the other voting groups.

In the more general situation when we only require $p_3 = 0$, we see that $(1, 0, 0)$ is still an equilibrium point. This means that the entire population may tend to become Susceptible Individuals if p_3, the rate at which individuals choose to become Moderates due to personal influence, equals zero. By computing the Jacobian matrix and evaluating it at the equilibrium point $(1, 0, 0)$, we obtain eigenvalues to determine the stability of the equilibrium point.

The eigenvalues of the jacobian matrix are:

$$\begin{aligned}
\lambda_1 &= 0, \\
\lambda_{2,3} &= -1/2\epsilon - 1/2p_9 + 1/2 \quad \pm 1/2(\epsilon^2 - 2\epsilon p_9 - 2\epsilon\beta + p_9^2 + 2 \quad p_9 + \quad ^2 + 4p_{10} \quad + 4p_{10}p_9)^{1/2},
\end{aligned}$$

We expect the zero-eigenvalue due to the presence of a redundant equation. The equilibrium point will be stable if the remaining two eigenvalues are negative. We see that $\lambda_2 < 0$ and $\lambda_3 < 0$ if $p_3 = 0$ and if:

$$P < 1 \text{ and } Q < 1$$

where

$$\begin{aligned}
P &= \frac{}{p_9 + p_4 + p_{10}} \text{ and} \\
Q &= \frac{p_9}{p_9 p_4 - p_{10}}
\end{aligned}$$

However we can show that P is a redundant condition. To do this we first assume that

$$Q = \frac{p_9}{p_9 p_4 - p_{10}} < 1.$$

Next, we consider

$$P = \frac{}{p_9 + p_4 + p_{10}} \geq 1.$$

Since, $P \geq 1$, we know that

$$\geq p_9 + p_4 + p_{10}.$$

Multiplying both sides of the equation by p_9 we have:

$$p_9 \quad \geq p_9(p_9 + p_4 + p_{10}),$$

since all parameters are positive real numbers. Thus,

$$p_9 \quad \geq p_9 p_4 - p_{10} \quad \text{i.e. } Q \geq 1.$$

But this contradicts our assumption that $Q < 1$. Hence if $Q < 1$ then $P < 1$ also. Since $Q < 1$ implies $P < 1$, we can show that $(1, 0, 0)$ will be stable if $p_3 = 0$ and $Q < 1$.

We note that we can re-write Q as

$$\frac{}{p_4} + \frac{p_{10}}{p_9 p_4} < 1$$

as well.

Interpretation of Stability Conditions

Now that we have conditions to ensure mathematical stability, we must relate these conditions to the political arena. If $p_3 = 0$, we know that individuals do not motivate themselves to leave the susceptible state and it is possible that the population will move toward the equilibrium condition $(1, 0, 0)$ in which no one in the entire population supports Democrats or Republicans.

Thus, if the goal of political activists is to insure some individuals are Moderate Democrats or Moderate Republicans then P must not hold. Mathematically we see that P does not hold if either $\quad > p_4$ or $p_{10} \quad > p_9 p_4$.
We assume that it is very difficult to change the value of the personal influence parameters because these parameters represent a person's motivation entirely independent from the influence of other individuals. Thus, increased or decreased campaigning will not have any effect on the values of the personal influence parameters. Hence, we assume that in the two month period before the presidential elections a political party cannot cause a significant change in the values of the personal influence parameters. Thus parties should focus on changing the parameters that represent the frequency of successful interactions.

Alternately, to ensure $\quad > p_4$, Moderates must be able to convert Susceptible Individuals at a higher rate that members choose to leave the Moderate Groups. Alternately, to ensure $p_{10} > p_7 p_4$, Fanatics need to have frequent, effective interactions with Susceptible Individuals. So political activists should focus on making either the Moderate or Fanatical members of their party more convincing to keep recruitment rates elevated.

the following figure confirms that under parameter values that meet the requirements above, $(1, 0, 0)$ is a stable equilibrium point.

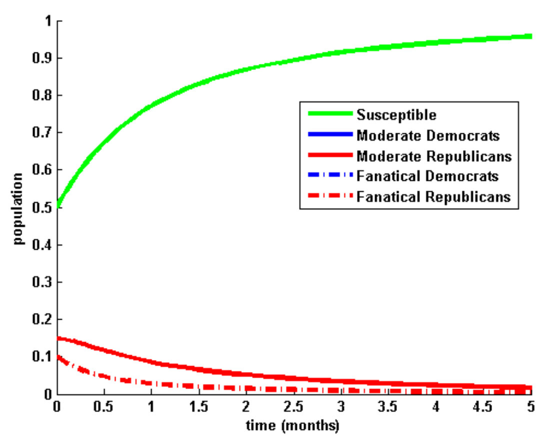


Figure 3: $b_1 = .5; b_2 = .4; b_3 = 1.9; b_4 = .5; b_6 = .25; b_7 = .3; b_8 = .25; c_9 = .75; c_{10} = .3; c_{12} = .25; g_1 = .1; g_2 = 1; p_3 = 0; p_4 = 2.9; p_9 = 3; p_{10} = .7$

3.1.2 Stability Analysis of the Equilibrium Point $E_{s_2} : (0, \frac{1}{2}, 0)$

We find that E_{s_2} is an equilibrium when $b_7 = g_1 = p_4 = p_{10} = 0$. The parameter b_7 represents the rate at which Moderates become Susceptible individuals due to interaction with other Moderate Individuals. The interaction represented by the b_7 term includes Moderates convincing other Moderates to change political alliances and Moderates irritating other Moderates which causes them to leave the political arena and become Susceptible Individuals. The parameter g_1 represents the rate at which Moderates become Fanatical Individuals due to interaction with other Moderates.

Furthermore, p_4 represents the rate at which personal influence causes Moderates to become Susceptible, and p_{10} represents the rate at which personal influence causes Moderates to become Fanatical individuals. So, we expect to see that the population will be evenly and entirely divided among the two Moderate classes when the aforementioned influences are not considered.

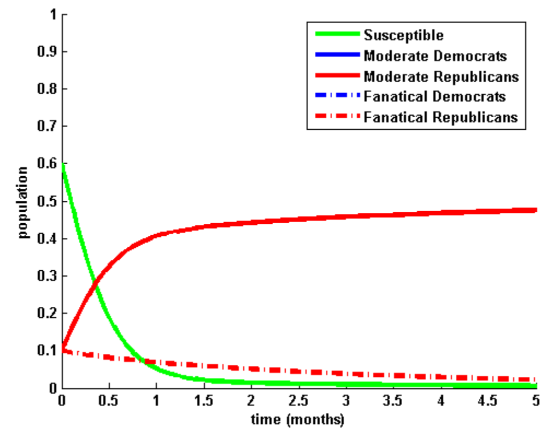


Figure 4: $b_1 = .5; b_2 = .4; b_3 = 1.9; b_4 = .5; b_6 = .7; b_8 = .2; c_1 = 0.5; c_2 = 0.3; c_3 = 1.0; c_9 = .75; c_{10} = .3; c_{12} = .25; g_2 = 1; p_3 = .6; p_9 = .5$

The Jacobian matrix evaluated at the equilibrium point again allows us to determine stability. The corresponding eigenvalues are:

$$\lambda_1 = 0, \lambda_2 = - \quad - 2p_3, \text{ and } \lambda_3 = \frac{1}{2}g_2 - \frac{1}{2}\rho - p_9$$

Accounting for the redundant equation allows us to state that this equilibrium point, (0,1/2,0) is stable if

$$b_7 = g_1 = p_4 = p_{10} = 0 \text{ and } \frac{g_2 + c_{12}}{c_9 + 2p_9} < 1. \tag{9}$$

Interpretation of Stability Conditions

When we relate this stability condition to politics, we see that if the goal is to have the entire population belong to the Moderate Class, then condition (9) must hold. To ensure (9) holds, we must have the following three conditions:

1. Moderates must not cause other Moderates to leave their political party.

2. Moderates cannot inspire themselves or other Moderates to become Fanatics.

3. Fanatics must become Moderates more frequently than Moderates become Fanatics.

In order to determine how well our system and this condition reflect reality, we would need to know real parameter values. This would tell us when or if it is reasonable to assume the above conditions occur.

3.1.3 Stability Analysis of the Equilibrium Point $E_{s_3} : (0, 0, \frac{1}{2})$

The next equilibrium condition we consider is when the population contains only Fanatical Individuals. Specifically, this equilibrium point is only an equilibrium point when

$c_{10} = p_9 = 0$. We notice that c_{10} represents the rate at which Fanatical Individuals become Moderate Individuals due to interaction with other Fanatical Individuals. This occurs when a Fanatical Individual convinces a Fanatical Individual of the opposing party to doubt their political views or when an interaction between two fanatical members of the same party causes one member to become less active. The parameter, p_9, represents the rate at which Fanatics become Moderates due to personal influence, which could be due to a lack of free time to invest in political activities. Hence when $c_{10} = p_9 = 0$, or when these types of influences never occur, the only way a Fanatical Individual can decide to give less support to his or her chosen candidate is to be influenced by a member of the Moderate Group.

The following equations are the symmetric system evaluated at the point $\left(0, 0, \frac{1}{2}\right)$. We again determine the stability of this solution by substituting the equilibrium point into the Jacobian matrix. The eigenvalues of this equilibrium are:

$$
\begin{aligned}
\lambda_1 &= 0, \\
\lambda_{2,3} &= \frac{-1}{4}(g_2 + {}_1 + 4p_3 + 2 \quad + 2\epsilon + \rho) \\
&\pm 1/4(-4 \quad g_2 + 4 \quad {}_1 + 4 \quad \rho + 4 \quad {}^2 + 8\epsilon\alpha \\
&-16p_{10} \quad + 8 \quad {}_1 p_3 + 16p_3^2 + 16p_3\epsilon + 8p_3\rho - 8p_3 g_2 - 32p_3 p_{10} + g_2^2 \\
&+2g_2 \quad {}_1 + 4g_2\epsilon - 2g_2\rho + \quad {}_1^2 + 4 \quad {}_1\epsilon - 2 \quad {}_1\rho + 16p_3 \quad + 4\epsilon^2 - 4\epsilon\rho + \rho^2)^{1/2}.
\end{aligned}
$$

The zero eigenvalue results from the constant population in our model. Hence, $(0, 0, \frac{1}{2})$ is stable when:

$$p_9 = c_{10} = 0 \text{ and } \frac{c_9}{c_{12} + g_2 + 2p_{10}} < 1. \tag{10}$$

Interpretation of Stability Conditions

An interpretation of this stability condition tells us that in order to have the entire population be Fanatical, i.e., be very active in politics, then (10) must hold.
If we want (10) to hold, then:

- Moderates must become Fanatics more frequently than Fanatics become Moderates.
- Fanatics must not drive other Fanatics to become Moderates. Also, personal influence must not cause Fanatics to become Moderates.

In other words, if the goal is to have the entire population be very active in politics, then the influences described previously that are associated with p_9 and c_{10} must not occur. Also, both the rate at which Fanatical Individuals convince Moderates to become active members of the Fanatic class, $(c_{12} + g_2)$, and the rate at which Moderates choose to become more active Fanatics, (p_{10}), must be high.

Specifically, it must be higher than the rate at which interaction between Moderates and Fanatics convinces the Fanatics to become less active Moderates themselves; these interactions include when Moderates convince Fanatical supporters of the opposing candidate to doubt their political views.

Figure 5 shows that under parameter values that meet the requirements above, $(0, 0, \frac{1}{2})$ is a stable equilibrium point.

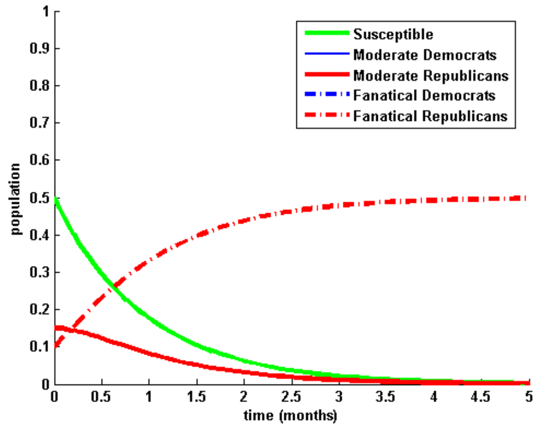


Figure 5: Here we have a numerical representation of a situation that meets the stability conditions for E_{s3}

3.1.4 Stability Analysis of the Equilibrium Point $E_{s_4} : f(V, A, g_1, \quad , \quad , p_3, g_2, \nu, \rho, c_{10}, p_9, \quad , \quad _1, \epsilon, p_{10})$

Finally, we will analyze the endemic equilibria, E_{s_4}. As usual, we can determine the stability of this equilibrium by linearizing about this solution. However, the procedure to find the eigenvalues of the Jacobian matrix is computationally difficult and the eigenvalues are too complex to reduce to a readable form.

3.2 Explanation of Cases

To further analyze the dynamics of our ordinary differential equations model we consider various additional cases where certain parameters in our system are ignored. We consider specific cases by eliminating parameters which helps in analyzing and interpreting the behavior of the system. In one case, we consider a situation where there is no decrease in political interest for any reason. This case only allows individuals to become more interested and supportive in politics. For example, the parameter b_6 represents a situation where a fanatical Democrat deters a Moderate Democrat from involving themselves in politics. This is considered a negative influence upon their own party. We can zero out these parameters to eliminate these negative influences, whether from within the party or from the opposing party. Thus mathematically we can set $b_6, b_{10}, b_{11}, b_{12}, c_7, c_3, c_{12}, d_1$ to zero. Not only are we considering the situation where an individual cannot become less interested in politics, but we also consider how counterproductive influences from the opposing party affect the system. This means that an individual fails to persuade another individual to become interested in their respective party. An example of a counterproductive influence in our model is b_1, where a Moderate Republican pushes an undecided or apathetic individual to the Moderate Democratic class. We zero out b_1 to eliminate this influence. We also set these parameters equal to zero to eliminate counterproductive influence: $b_2, b_4, g_3, g_4, b_{16}, b_{17}, c_9, c_{10}, g_1, g_2, d_3, d_4$.

In other cases, we do not take into consideration personal influence when an individual from either of the Moderate Groups decides to switch their political affiliation. Recall that personal influence includes an individual's pre-existing ideas and opinions based on their up-bringing or their experiences in previous events that a ect their political decisions. We denote these personal influence parameters as: $p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{12}, p_{13}$. Thus, in some cases we assume an individual in a given class can only decide to change their political affiliation based on the influence imposed when interacting with other individuals.

4 Strong Political Interest

Clean Elections are measures that states can choose to implement which set up a reserve of funds that candidates have the option to use. The only restriction is that these politicians must sign an agreement that forces them to never use additional funds (including corporate funds) in their election campaign. Some states, such as Arizona, New Mexico, and California, have either considered Clean Elections or already have them implemented in their respective political system. Many individuals feel that Clean Elections should be taken a step further and include measures that would not allow for individuals to use any smear tactics in their campaign. This would reduce the rate of negative influence between individuals drastically, almost making it negligible. If this measure were implemented, what impact would this have on what information is spread to individuals in the political system? If we assume that negative campaigning is what leads individuals to dislike the candidate who is running the campaign, and that negative influences from Moderates is minimal enough that we can exclude it, then we can qualitatively analyze this scenario in our model.
As we qualitatively analyze our mathematical model, we consider the situation where there is no decreasing political interest and no counterproductive influences from opposing parties. We first examine the interactions that occurred between the Moderate Democrats and the Moderate Republicans in the general case. All interactions that led an individual to go to the opposite party of the person they interacted with must be zeroed out. When we continue this process of removing negative interactions throughout the entire model, we obtain $b_6 = b_7 = b_8 = p_5 = c_9 = c_{10} = p_9 = d_3 = d_4 = p_{13} = b_{10} = b_{11} = b_{12} = p_4 = 0$ and $b_1 = b_2 = g_3 = g_4 = b_{16} = b_{17} = g_1 = g_2 = 0$. This prevents the members of any Moderate Class from joining the Susceptible Group. Thus an individual is only allowed to become more involved in a given party and any interactions with a given party have positive result.

The relevant equilibria to this system are

$$(S^*, E_D^*, E_R^*, F_D^*, F_R^*) = (0, 0, 0, x, 1 - x), \tag{11}$$

where x is a specific proportion that changes depending on the parameters. In terms of our application, this corresponds to the case when all voters end up as Fanatics. This is expected, given that once an individual commits to a political party we assume that they can only become more committed to that affiliation.

We compute the Jacobian of the system and evaluate it at our given equilibria 11 to determine stability. Three of the eigenvalues determined from this are:

$$\lambda_1 = -b_4 F_D - p_6 - b_{15} F_R - p_3, \qquad \lambda_2 = 0, \quad \text{and} \quad \lambda_3 = 0,$$

which are never positive. The other two eigenvalues are quite complicated, but have the form:

$$\lambda_{4,5} \quad = \quad f(c_7, F_D, F_R, p_7, p_{12}, p_8, p_{10}, d_1, c_2, c_{12}, c_6, c_3),$$

Examining the characteristic polynomial shows that the roots cannot have opposite sign nor can they change sign. We observe that both λ_4 and λ_5 are always negative.

Since three out of five of our eigenvalues are negative and two are zero, equilibria (11) are considered center-stable. Thus, we see that if we do not have counterproductive influences and we do not allow for a loss in political interest, we have the situation where everyone will become members of one of the Fanatical Groups. (As discussed earlier, one of the zero eigenvalues is expected because of the redundant equation.)

We consider some further simplifications of this case which possess the same relevant stable equilibria and the same stability conditions in the hopes of finding additional equilibria that may exist under more restrictive situations. We still consider all the parameters that were zeroed out in the main case above.

- Case 1: Moderate Individuals May Switch Allegiance Due to Disillusionment with Own Party

In this case there is no positive influence from E_D on E_R nor from E_R on E_D. No positive influences between groups implies that no member from the opposing group can influence the other to join that party. Thus, Moderate Democrats and Moderate Republicans do not have any influence on each other, and so we set $c_5 = c_6 = c_1 = c_2 = 0$. Only the Fanatics in each party are considered to have the ability to influence individuals to move in between the two Moderate Groups. In addition, individuals can choose to leave their party on their own without any interaction with individuals from other groups.

- Case 2: No Positive Interaction and No Personal Influence Between Moderate Classes

Once an individual is in a Moderate Group he or she cannot move directly to the opposing Moderate Group due to personal influence. In our application, this means there is no personal influence from E_D on E_R and from E_R on E_D. This assumption mathematically means $p_7 = p_8 = 0$. We also consider no positive interaction from E_D on E_R and from E_R on E_D, which is represented by: $c_5 = c_6 = c_1 = c_2 = 0$.

- Case 3: No Counterproductive Influence on Either Voting Party

In this situation, we consider no negative influence from an individual's own party, which mathematically can be represented as $c_7 = c_3 = 0$. This means that fanatical Democrats and Republicans do not have any negative influence among their respective Moderate groups.

- Case 4: No Negative Interaction From Own Party and No Personal Influence

This situation has the same conditions displayed for Case 3, but additionally disallows personal influence between E_D and E_R, which means we set $p_7 = p_8 = 0$. In this case, the Fanatical Democrats and Republicans do not have any influence on each other and there is no personal influence that would inspire an individual to move from one Moderate group to another Moderate group.

- Case 5: No Negative Influence From Own Party and No Positive Influence Between Moderate Classes

We assume that there is no positive influence from E_D on E_R and from E_R on E_D, hence in our mathematical model we have $c_5 = c_6 = c_1 = c_2 = 0$. In this situation, the only interaction between E_D and E_R is due to personal influence.

- Case 6: No Interaction and Personal Influence between Moderate Classes

Here we assume there is no interaction between E_D and E_R, which can be expressed as $c_5 = c_6 = c_1 = c_2 = 0$. We also consider no personal influence between E_R and E_D, which can be mathematically expressed as $p_7 = p_8 = 0$. These conditions imply that individuals from either the Moderate Democrat or Moderate Republican Group cannot move directly to the opposing party. Once an individual is in a Moderate Group, that individual can either remain in that group or progress to the Fanatical Class.

Earlier, it was mentioned that all of the subcases above have the same relevant equilibria, which are stable under the same conditions. For all of the subcases the first three eigenvalues that we obtain are always non-positive. The last two of the eigenvalues of the equilibria that we obtain are negative for all cases and are complicated, except Case 6. For this subcase, the last two eigenvalues are simpler to analyze than the other previously mentioned cases. They are:

$$\lambda_4 = -c_{12}x - p_{10} \qquad \lambda_5 = -d_1(1-x) - p_{12}$$

Both of these eigenvalues are always negative as well. So, we find that in all subcases above, we only have the equilibria (11), which is

$$(S^*, E_D^*, E_R^*, F_D^*, F_R^*) = (0, 0, 0, x, 1-x),$$

where again x is a specific proportion that changes depending on the parameters. Given that all of the eigenvalues that are found are negative or zero, we conclude that the equilibrium point is center-stable.

As previously mentioned in Case 6, we obtain eigenvalues that are simpler to analyze than the two eigenvalues of the other subcases considered. This is no surprise since more parameters are set equal to zero compared to the other subcases. For this subcase, though the general conditions remain the same, all the parameters between E_D and E_R are eliminated, meaning that once an individual affiliates themselves with a political party, they remain in that party and cannot change their political party over time. Since, we are only concerned with forward movement with no possibility for an individual to return to a less active class or change political stances, it makes sense that in a simpler system the resulting eigenvalues are easier to analyze.

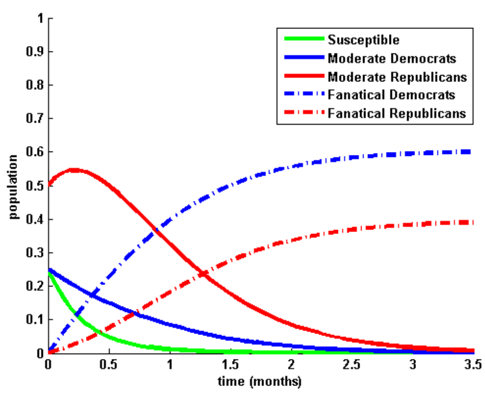


Figure 6: $b3 = 3; b4 = .4; b14 = 4; b15 = 1.5; c12 = 3; d1 = 2.5; p3 = .2; p6 = .3; p7 = 1; p8 = 2; p10 = 2; p12 = .2$

In all of the subcases that we considered from the set conditions it is expected that over time, individuals move to the Fanatical Classes. This is verified when we find out that the equilibria from $(0, 0, 0, x, 1 - x)$ is center-stable. We can conclude that with this equilibria that the equilibra remains the same and still maintain stability, no matter what influential parameters are eliminated between E_D and E_R.

As shown by the numerical solution in Figure (6), if the negative influential parameters are eliminated, there is nothing that discourages an individual from supporting a particular presidential candidate. In present day politics, if an individual becomes a Fanatic, they will most likely remain in this group based on the their strong support for their candidate and political party. It is extremely unlikely that this individual would be less supportive in a two month period and would no longer support that candidate.

5 Decrease in Political Interest Only Due to Personal Influence

We choose to investigate this case because it is representative of a population of individuals who are not easily swayed from their initial beliefs. Here, we ignore the effect of negative influence in order to explore the possibility that political campaigns can be successful without smear tactics, which commonly result in negative influences. Furthermore, we assume that people choose to support a candidate they agree with more frequently than they choose to support a candidate because they disagree with the opposing candidate. These assumptions allow us to consider the negative influences negligible.

Thus we focus on the situation when backward movement is only due to personal influence and no negative influence from the opposing party voters is considered. Mathematically this implies that we have $b_6 = b_7 = b_8 = c_9 = c_{10} = d_3 = d_4 = b_{10} = b_{11} = b_{12} = 0$ and $b_1 = b_2 = g_3 = g_4 = b_{16} = b_{17} = g_1 = g_2 = 0$. As we stated earlier, when we say backward movement we mean that individuals transition from a more active to a less active state. Recall that personal influence is represented by terms that contain p_i parameters. Once again, negative influence refers to an interaction that causes a person to move away from the party of the person with whom they are interacting. This case is different from the previous case since it allows for backward movement due to personal influence whereas the last case did not allow backward movement at all. In other words, p_4, p_5, p_9, and p_{13} are now non-zero. Since individuals cannot be externally convinced to leave their group, they can only move backward due to personal preferences. This can be interpreted to mean that individuals are more strongly-affiliated with their chosen group than in the general case because individuals cannot be externally convinced to leave. However, in comparison to the previous case we see that Fanatical Individuals are allowed to be less strongly affiliated with their group because now backward movement is allowed. Also, we allow a person to transition directly from the Moderate Democrat Class to the Moderate Republican class and vice versa.

We have one nontrivial relevant equilibrium point:

$$(S^*, E_D^*, E_R^*, F_D^*, F_R^*) = \left(\frac{N^- + N^+}{D^- + D^+}, B, Z, \frac{p_{10}B}{Bc_{12} - p_9}, \frac{p_{12}Z}{-d_1 Z + p_{13}}\right), \Bigg($$

which we have written in this form due to its algebraic complexity. We find that B is the root of a fourth degree polynomial with coefficients that are parameter based,

$$\begin{aligned}
N^- &= -Zd_1B^2c_{12}p_5 - Z^2d_1Bc_{12}p_4 - p_{13}p_9p_5B - p_{13}p_9p_4Z, \\
N^+ &= Z^2d_1p_9p_4 + Zd_1p_9p_5B + p_{13}B^2c_{12}p_5 + p_{13}Bc_{12}p_4Z, \\
D^- &= -Zd_1B^2c_{12}b_3 - Zd_1Bc_{12}p_6 - Z^2d_1Bc_{12}b_{14} - Zd_1Bc_{12}p_3 - p_{12}Zb_{15}p_9 - \\
&\quad p_{13}b_4p_{10}B - p_{13}p_9b_3B - p_{13}p_9p_6 - p_{13}p_9b_{14}Z - p_{13}p_9p_3, \\
D^+ &= Zd_1b_4p_{10}B + Zd_1p_9b_3B + Zd_1p_9p_6 + Z^2d_1p_9b_{14} + Zd_1p_9p_3 + \\
&\quad p_{12}Zb_{15}Bc_{12} + p_{13}B^2c_{12}b_3 + p_{13}Bc_{12}p_6 + p_{13}Bc_{12}b_{14}Z + p_{13}Bc_{12}p_3,
\end{aligned}$$

and $Z = E_R^*$ where Z is defined implicitly, by

$$S^* + E_D^* + Z + F_D^* + F_R^* = 1.$$

There are two conditions for S^* that will ensure that S^* is positive:

$$|D^-| > D^+ \text{ and } |N^-| > N^+,$$

or

$$|D^-| < D^+ \text{ and } |N^-| < N^+.$$

We also need $B > 0$ in order to have a positive E_D^* value. Also, it must be the case that $p_9 > c_{12}B$ in order to have a positive F_D^* value. We also will need $p_{13} > d_1Z$ in order to have a positive F_R^* value. Finally, we must also clearly have S^*,E_D^*, E_R^*, F_D^*, and F_R^* to be non-negative.

Finding the eigenvalues of the jacobian matrix evaluated at the equilibrium point listed above without specific parameters values proved elusive. In hopes of gaining some insight into the dynamics of this case, we set all non-zero parameters equal to 1 for simplicity. The following is the only relevant equilibrium point given by the system for these parameters, and it is given as $(S^*, E_D^*, E_R^*, F_D^*, F_R^*)$:

$$(0.135, 0.193, 0.193, 0.239, 0.239)\,.$$

The resulting nonzero eigenvalues for the equilibrium above are:

$$\lambda_1 = -5.321, \lambda_2 = -1.492, \lambda_3 = -4.284, \lambda_4 = -.548$$

Hence for these parameter values the equilibrium values is stable. Thus, over time a population will tend toward some stable state where a certain percentage of the population holds the views associated with each group.

Interpretation of Stability

Recall that we assume that it is very difficult to change the value of the personal influence parameters. With this in mind, we see we can restate the stability conditions that must hold in order to ensure that the population does not tend to consist entirely of Susceptible individuals more simply. First, we find that b_{14} (the rate at which Moderate Republicans convince Susceptible Individuals to become Moderate Republicans) needs to be large and in particular it needs to be much larger than p_4 (the rate at which personal influences cause Moderate Republicans to become Susceptible individuals). Similarly, b_3 (the rate at which Moderate Democrats convince Susceptible Individuals to become Moderate Democrats needs to be large and in particular it needs to be much larger that p_5 (the rate at which personal influences cause Moderate Democrats to become Susceptible Individuals). These results seem to be common sense; they tell us that individuals must join the two main parties with higher frequency than individuals lose interest in these parties. However, it is interesting to note that mathematically the stability conditions do not include any component related to Fanatical Individuals. This situation (when we disallow personal influence as a reason for switching allegiance and decrease in political interest is only due to personal influence) indicates that the rate at which Fanatics influence and interact with others does not significantly influence a political party's recruitment rate. We find this result to be surprising since politicians spend a lot of time and money on advertising during a presidential campaign and in our model we classify this type of interaction as a Fanatical Interaction. Further investigation of this system of real life parameter values could clarify why we have found this result.

We also notice that if we add the condition that there is no negative influence from one's own party ($c_7 = c_3 = 0$) to the conditions in this case, then we obtain the same IFE with the same conditions on center stability. From this we can conclude that negative influence from an individual's own party does not have an effect on the IFE and its stability.

5.1 Personal Influence Cannot Cause a Switch of Allegiances Between Moderate Members

To further qualitatively analyze our mathematical model, we remain in the situation where backward movement is only due to personal influence. Now, however, we set additional parameters in our system equal to zero. In this case, we focus on what happens when backward movement is only due to personal influence and no negative influence from the opposing party voters is considered. In addition, we assume there is no positive influence from E_D on E_R, there is no positive influence from E_R on E_D, and there is no personal influence in the forward movement. Mathematically this means that $b_6 = b_7 = b_8 = c_9 = c_{10} = d_3 = d_4 = b_{10} = b_{11} = b_{12} = 0$, $b_1 = b_2 = g_3 = g_4 = b_{16} = b_{17} = g_1 = g_2 = 0$, $c_1 = c_2 = 0$, $c_5 = c_6 = 0$, and $p_6 = p_3 = p_{10} = p_{12} = 0$ respectively. This subcase differs from the previous case because it reduces the interactions that cause an individual to transition between the E_D and E_R classes and it assumes a person cannot become more fanatical due to personal influence. Thus we emphasize the role that interaction has on influencing a person to become more fanatical. We are interested in analyzing this case because we hope to investigate variations in the way individuals switch party allegiance (transition between E_R and E_D) and how interactions with others (ignoring personal motivation as a means of becoming fanatical) leads to changes in political affiliations during the short period before a presidential election.

Under these assumptions our model gives rise to an Idea Free Equilibrium (IFE) of:

$$(S^*, E_D^*, E_R^*, F_D^*, F_R^*) = (1, 0, 0, 0, 0).$$

As previously explained, the IFE is an equilibrium in which the entire population is in the Susceptible group. Since Susceptible Individuals do not vote for neither the Republican nor Democratic candidates then the IFE is when no one in the population is affiliated with one of our two major political parties.
The eigenvalues for the IFE are:

$$\lambda_1 = 0, \quad \lambda_2 = -p_9, \quad \lambda_3 = -p_{13},$$

$$\begin{aligned} \lambda_{4,5} \;=\; & \frac{1}{2}((-p_5 + b_3 - p_4 - p_8 + b_{14} - p_7) \pm ((-p_5 + b_3 - p_4 - p_8 + b_{14} - p_7)^2 + \\ & (-4b_{14}b_3 - 4p_5p_7 - 4p_4p_5 - 4p_4p_8 + 4b_3p_7 + 4p_4b_3 + 4b_{14}p_5 + 4b_{14}p_8))^{1/2} \end{aligned}$$

The eigenvalues indicate that the equilibrium point, $(1, 0, 0, 0, 0)$, is stable if:

$$-p_5 - p_7 + b_{14} + b_3 - p_8 - p_4 < 0 \quad \text{and} \quad -b_{14}b_3 - p_5p_7 - p_4p_5 - p_4p_8 + b_3p_7 + p_4b_3 + b_{14}p_5 + b_{14}p_8 < 0$$

Or equivalently, if

$$\frac{b_{14}+b_3}{p_5+p_7+p_8+p_4} < 1 \quad \text{and} \quad \frac{p_7}{b_{14}-p_4}+\frac{p_8}{b_3-p_5} < 1$$

This means that if the goal is to have potential voters support the two main political parties, then these parties need to ensure that these stability conditions are not met. We continue to assume that it is very difficult to change the value of the personal influence parameters because these parameters represent a person's motivation entirely independent from the influence of other individuals. As a result, increased or decreased campaigning will not have any effect on the values of the personal influence parameters. We assume that in the two month period before the presidential elections a political party cannot cause a significant change in the values of the personal influence parameters and so we focus on the parameters that represent the frequency of successful interactions.

With this in mind, we see that in order to ensure that the population does not consist entirely of Susceptible Individuals b_{14} (the rate at which Moderate Republicans convince Susceptible Individuals to become Moderate Republicans) needs to be large and in particular it needs to be much larger than p_4, which is the rate at which personal influences cause Moderate Republicans to become Susceptible Individuals. Similarly, b_3 (the rate at which Moderate Democrats convince Susceptible Individuals to become Moderate Republicans) needs to be very large and in particular it needs to be much larger that p_5, which is the rate at which personal influences cause Moderate Democrats to become Susceptible Individuals. These results seem to be common sense; however, it is interesting to note that mathematically the stability conditions do not include any component related to Fanatical Individuals. This indicates that the rate at which Fanatical individuals cause others to change classes is not important in regard to voter turn out in this situation when we do not allow personal influence as a reason for Moderates switching allegiance.

We notice that if we add the condition that there is no negative influence from your own party ($c_7 = c_3 = 0$) to the conditions in this case, then we obtain the same IFE with the same conditions on center stability. From this we can conclude that negative influence from an individual's own party does not have an effect on the IFE and its stability.

5.2 Personal Influence Can Only Decrease Political Interest

Now, consider a new subcase where certain additional parameters in our system are ignored. We still consider what happens in our system when backward movement is only due to personal influence, there is no negative influence from the opposing party voters nor from ones' own party, there is no personal influence on the forward movement, there is no positive influence from E_D on E_R (or vice versa). Additionally, now we assume personal influence cannot cause an individual to move between the E_R and E_D classes. Mathematically this means that $b_6 = b_7 = b_8 = c_9 = c_{10} = d_3 = d_4 = b_{10} = b_{11} = b_{12} = 0$, $b_1 = b_2 = g_3 = g_4 = b_{16} = b_{17} = g_1 = g_2 = 0$, $c_7 = c_3 = 0$, $c_5 = c_6 = 0$, $p_6 = p_3 = p_{20} = p_{12} = 0$, $c_1 = c_2 = 0$, and $p_7 = p_8 = 0$ respectively.

As we stated earlier, backward movement means that individuals transition from a more fanatical state to a less fanatical state. Recall that personal influence is represented by terms that contain p_i parameters. Once again, negative influence refers to an interaction that causes a person to move away from the party of the person with whom they are interacting. Furthermore, we are assuming that an individual cannot transition directly from a Moderate Democrat to a Moderate Republican or vice versa under any circumstances. Instead an individual must again become a member of the Susceptible group before becoming a Moderate in support of his or her former rivals. This means that we set $c_1 = c_2 = c_3 = c_5 = c_6 = c_7 = p_7 = p_8 = 0$. We are considering this case because we are interested in investigating how the presidential election will be a ected if individuals undergo a lengthy thought process before switching political affiliations.

The equilibria $(S^*, E_D^*, E_R^*, F_D^*, F_R^*)$ in this case are:

$$E_1 = (1,0,0,0,0),\ E_2 = \left(\frac{p_4}{b_{14}}, 0, 1-\frac{p_4}{b_{14}}, 0, 0\right),\ E_3 = \left(\frac{p_5}{b_3}, 1-\frac{p_5}{b_3}, 0, 0, 0\right),$$
$$E_4 = \left(W, 0, \frac{p_{13}}{d_1}, 0, \frac{-p_{13}(Wb_{14}-p_4)}{(Wb_{15}d_1)}\right),\ E_5 = \left(W, \frac{p_9}{c_{12}}, 0, \frac{-p_9(Wb_3-p_5)}{(Wb_4c_12)}, 0\right),$$
$$E_6 = \left(\frac{p_4}{b_{14}}, \frac{p_9}{c_{12}}, Z, \frac{-p_9(p_4b_3-p_5b_{14})}{(p_4b_4c_{12})}, 0\right),\ E_7 = \left(\frac{p_5}{b_3}, Y, \frac{p_{13}}{d_1}, 0, \frac{p_{13}(p_4b_3-p_5b_{14})}{(p_5b_{15}d_1)}\right),$$
$$\text{and } E_8 = \left(W, \frac{p_9}{c_{12}}, \frac{p_{13}}{d_1}, \frac{-p_9(Wb_3-p_5)}{(Wb_4c_{12})}, \frac{-p_{13}(Wb_{14}-p_4)}{(Wb_{15}d_1)}\right).$$

where W, Y, and Z are real numbers such that $W = 1 - E_D^* - E_R^* - F_D^* - F_R^*,// \ Y = 1 - S^* - E_R^* - F_D^* - F_R^*$, and $Z = 1 - S^* - E_D^* - F_D^* - F_R^*$. We may perform this simplification because the total population of the system N is constant and $N = 1 = S + E_D + E_R + F_D + F_R$. At this time, we have chosen to only analyze a subset of the equilibrium points, E_1, E_2, E_3, and E_6. In each of the following four subsections of this paper, each of these selected equilibria will be analyzed for stability conditions.

In this case the stability conditions tell us that, when the IFE is stable, the rate at which Moderate Democrats choose to join the Susceptible Group due to personal influence, p_5, is higher than the rate at which Moderate Democrats convince Susceptible Individuals to become Democratic Voters, b_3, via interaction. The analogous statement holds for the respective Republican Classes. Under these conditions the entire population of the system will eventually become Susceptible individuals and will not vote Democrat or Republican.

During our analysis, we also considered non-zero values for c_1, c_2, c_5, and c_6 and obtained the IFE with the same eigenvalues and stability conditions as E_1. Since c_1, c_2, c_5, and c_6 represent the positive influence from Moderate Democrats (E_D) on Moderate Republicans (E_R) and the positive influence from Moderate Republicans, E_R, on Moderate Democrats, E_D, then we see that this positive influence does not affect whether a population will decide collectively not to vote if the conditions for stability are met.

While understanding the conditions that will ensure the population consists entirely of non-voters is interesting, it is more useful for political analysts to know what can be done to ensure that the IFE is unstable because this will increase the number of individuals that support the Democratic and Republican Parties. In order to avoid the stability conditions described above and increase the number of voters that choose Democratic or Republican candidates then the Democratic Party should increase ensure that Moderate Democrats recruit individuals into the party more frequently than individuals decide to leave for personal reasons. Alternatively (or additionally) the Republican Party should make an effort to have Moderate Republicans recruit individuals into the party at a higher rate than members leave for personal reasons. As explained in section 5.1, we assume that during the two month period before the presidential elections political parties cannot significantly influence the value of the personal influence parameters and that is why we focus on the effect manipulating the interaction parameters will have on the election. Also, it is interesting to note that under our assumptions this interpretation is almost the same as the one in the previous section. Since the only difference in the two cases is that we do assume personal influence does not cause individuals to switch Moderate Groups in this case, we see that this type of personal influence does not greatly affect the IFE. Once again we see the surprising result that Fanatical Interactions do not have a significant influence on whether the population consists of Democrats and Republicans. Hence, this is another case that we would like to investigate with confirmed parameter values.

Interpretation of Stability

In this case the stability conditions tell us what conditions are necessary for the Republicans to win the presidential election with all of their supporters classified as Moderate Republicans. Once again we have the condition that Specifically, the rate at Susceptible Individuals are convinced to join the Republican party by Moderate Republicans, (b_{14}), must be higher than the rate at which Moderate Republicans leave the party due to personal influences (p_4). Additionally, the last two stability conditions indicate that the rate at which Moderate Republicans recruit former Democrats must be greater than the rate at which Moderate Democrats woe former Republicans to join their party. Simultaneously, the rate at which individuals transition through the Moderate Republican Group must be higher than the rate at Moderates become Fanatics. This case is interesting because while it is the first in this section to suggest that the Fanatical Groups play a role (albeit a relatively small one) in determining the outcome of the election under the conditions discussed in this section.

5.2.1 Stability Analysis of the Equilibrium Point $E_1 : (1, 0, 0, 0, 0)$

The first equilibrium we will analyze is the IFE, E_1. The eigenvalues of this equilibrium are:

$$\lambda_1 = 0, \quad \lambda_2 = b_3 - p_5, \quad \lambda_3 = b_{14} - p_4, \tag{12}$$

$$\lambda_4 = -p_9, \quad \lambda_5 = -p_{13}. \tag{13}$$

These eigenvalues will all be negative and so the equilibrium will be stable if $p_5 > b_3$ and

$p_4 > b_{14}$. In this case the stability conditions tell us that, when the IFE is stable, the rate at which Moderate Democrats choose to join the Susceptible group due to personal influence, p_5, is higher than the rate at which Modern Democrats convince Susceptible individuals to become Moderate Democrats, b_3, via interaction. The analogous statement holds for the respective Republican classes. Under these conditions the entire population of the system will eventually become Susceptible Individuals and will not vote Democrat or Republican.

During our analysis we also consider non-zero values for c_1, c_2, c_5, and c_6, and obtain the IFE with the same eigenvalues and stability conditions as E_1. Since c_1, c_2, c_5, and c_6 represent the positive influence from Moderate Democrats, (E_D), on Moderate Republicans, (E_R), and the positive influence from Moderate Republicans on Moderate Democrats, then we see that this positive influence does not affect whether a population will decide collectively not to vote Democrat or Repbulican if the conditions for stability are met.

Interpretation of Stability Conditions

In this case the stability conditions tell us that when the IFE is stable, the rate at which Moderate Democrats choose to join the Susceptible Group due to personal influence, p_5, is higher than the rate at which Moderate Democrats convince Susceptible Individuals to become Democratic Voters, b_3, via interaction. The analogous statement holds for the respective Republican Classes. Under these conditions the entire population of the system will eventually become Susceptible individuals and will not vote Democrat or Republican.

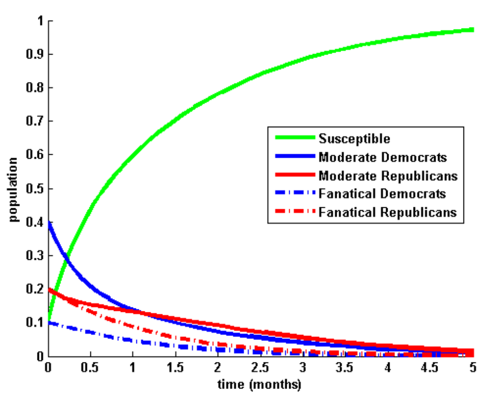


Figure 7: $b_3 = 1; b_4 = 2; b_{14} = 1; b_{15} = 1.4; c_{12} = 1; d_1 = 1; p_3 = 0; p_4 = 2; p_5 = 2; p_7 = 10; p_9 = 1; p_{13} = 1$

During our analysis, we also considered non-zero values for c_1, c_2, c_5, and c_6 and obtained the IFE with the same eigenvalues and stability conditions as E_1. Since c_1, c_2, c_5, and c_6 represent the positive influence from Moderate Democrats (E_D) on Moderate Republicans (E_R) and the positive influence from Moderate Republicans, E_R, on Moderate Democrats, E_D, then we see that this positive influence does not affect whether a population will decide collectively not to vote if the conditions for stability are met.

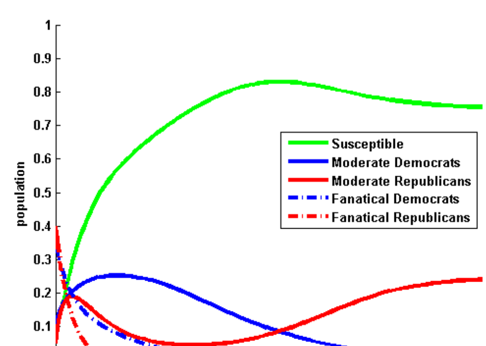
While understanding the conditions that will ensure the population consists entirely of non-voters is interesting, it is more useful for political analysts to know what can be done to ensure that the IFE is unstable because this will increase the number of individuals that support the Democratic and Republican Parties. In order to avoid the stability conditions described above and increase the number of voters that choose Democratic or Republican candidates then the Democratic Party should increase ensure that Moderate Democrats recruit individuals into the party more frequently than individuals decide to leave for personal reasons. Alternatively (or additionally) the Republican Party should make an effort to have Moderate Republicans recruit individuals into the party at a higher rate than members leave for personal reasons. As explained in section 5.1, we assume that during the two month period before the presidential elections political parties cannot significantly influence the value of the personal influence parameters and that is why we focus on the effect manipulating the interaction parameters will have on the election. Also, it is interesting to note that under our assumptions this interpretation is almost the same as the one in the previous section. Since the only difference in the two cases is that we do assume personal influence does not cause individuals to switch Moderate Groups in this case, we see that this type of personal influence does not greatly affect the IFE. Once again we see the surprising result that Fanatical Interactions do not have a significant influence on whether the population consists of Democrats and Republicans. Hence, this is another case that we would like to investigate with confirmed parameter values.

5.2.2 Stability Analysis of the Equilibrium Point E_2: $(\frac{p_4}{b_{14}}, 0, 1 - \frac{p_4}{b_{14}}, 0, 0)$

The second equilibrium we will analyze is one in which the population divides itself into Susceptible Individuals and Moderate Republicans, E_2. We see that $b_{14} \geq p_4$ is needed to have a relevant equilibrium point. The eigenvalues of this equilibria are:

$$\lambda_1 = 0, \quad \lambda_2 = p_4 - b_{14}, \quad \lambda_3 = \frac{p_4 b_3 - p_5 b_{14}}{b_{14}}, \quad \lambda_4 = -p_9, \quad \lambda_5 = \frac{-d_1 p_4 + d_1 b_{14} - b_{14} p_{13}}{b_{14}}$$

Thus, the equilibrium will be stable if $p_4 < b_{14}$, $p_4 b_3 < p_5 b_{14}$, and $d_1 b_{14} < d_1 p_4 + b_{14} p_{13}$. In this case the stability conditions tell us what conditions are necessary for the Republicans to win the presidential election with all of their supporters classified as Moderate Republicans. Specifically, the rate at which individuals are influenced by Moderate Republicans to leave the Susceptible group and become Moderate Republicans (b_{14}) must be higher than the rate at which Moderate Republicans choose to join the susceptible group (p_4) for the population to stabilize at the equilibrium point. Additionally, the last two stability conditions indicate that the rate at which voters move from the Democratic Party to the Republican Party due to the influence of the Moderate groups must be greater than the rate at which voters move from the Republican Party to the Democratic Party due to the influence of the Moderate groups, while the rate at which people leave the Moderate Republicans combined with the rate at which people enter the Moderate Republican group must be higher than the rate at which people enter the Fanatical Republican group. This case is interesting because it suggests that the fanatical groups play a relatively small role in determining the outcome of the election under the conditions discussed in this section.



Interpretation of Stability Condition

In this case the stability conditions tell us what conditions are necessary for the Republicans to win the presidential election with all of their supporters classified as Moderate Republicans. Once again we have the condition that Specifically, the rate at Susceptible Individuals are convinced to join the Republican party by Moderate Republicans, (b_{14}), must be higher than the rate at which Moderate Republicans leave the party due to personal influences (p_4). Additionally, the last two stability conditions indicate that the rate at which Moderate Republicans recruit former Democrats must be greater than the rate at which Moderate Democrats woe former Republicans to join their party. Simultaneously, the rate at which individuals transition through the Moderate Republican Group must be higher than the rate at Moderates become Fanatics. This case is interesting because while it is the first in this section to suggest that the Fanatical Groups play a role (albeit a relatively small one) in determining the outcome of the election under the conditions discussed in this section.

5.2.3 Stability Analysis of the Equilibrium Point E_3: $(\frac{p_5}{b_3}, 1 - p_5 b_3, 0, 0, 0)$

The third equilibrium point we will analyze is one in which the population divides itself into Susceptible individuals and Moderate Democrats, E_3.

We see that $b_3 \geq p_5$ is needed to have a relevant equilibrium. The eigenvalues of this equilibrium are:

$$\lambda_1 = 0,\ \lambda_2 = -b_3 + p_5,\ \lambda_3 = \frac{(p_5 b_{14} - p_4 b_3)}{b_3},\ \lambda_4 = \frac{(c_{12} b_3 - c_{12} p_5 - b_3 p_9)}{b_3},\ \lambda_5 = -p_{13}$$

Thus, the equilibrium will be stable if $p_5 < b_3$, $p_5 b_{14} < p_4 b_3$, and $c_{12} b_3 < c_{12} p_5 + b_3 p_9$. This case can be interpreted in the same manner as the previous case except that where Republicans are referenced in the previous case we refer to Democrats in this situation.

5.2.4 Stability Analysis of the Equilibrium Point E_6: $(S^*, E_D^*, E_R^*, F_D^*, F_R^*) = \left(\frac{p_4}{b_{14}}, \frac{p_9}{c_{12}}, Z, \frac{-p_9(p_4 b_3 - p_5 b_{14})}{(p_4 b_4 c_{12})}, 0\right)\Bigg($

The sixth equilibrium point we will analyze is one in which the population does not contain Fanatical Republicans. We observe that in this case the equilibrium point could contain some negative components. Only an equilibrium point with entirely positive values is relevant to the political situation we are investigating since populations cannot be negative. So we must consider the conditions that make all the components of the equilibrium positive.

Since $N = S + E_D + E_R + F_D + F_R$ and N, the total population is constant, then $Z = E_R^* = 1 - S^* - E_D^* - F_D^* - F_R^*$. Thus,

$$Z = E_R^* = \frac{(-p_4^2c_{12}b_4 - p_9b_{14}p_4b_4 + p_9b_{14}p_4b_3 - p_9b_{14}^2p_5 + b_{14}c_{12}p_4b_4)}{(b_{14}c_{12}p_4b_4)}$$

.

We conclude that E_R will be a positive quantity if:

$$-p_4^2c_{12}b_4 - p_9b_{14}p_4b_4 + p_9b_{14}p_4b_3 - p_9b_{14}^2p_5 + b_{14}c_{12}p_4b_4 \quad > \quad 0$$

i.e.

$$p_4^2c_{12}b_4 + p_9b_{14}p_4b_4 + p_9b_{14}^2p_5 \quad < \quad p_9b_{14}p_4b_3 + b_{14}c_{12}p_4b_4.$$

Furthermore, F_D^* will be a positive quantity if $p_9p_5b_{14} > p_9p_4b_3$.

We find that two eigenvalues of this equilibrium are $\lambda_1 = 0, \lambda_2 = Zd_1 - p_{13}$. The remaining three eigenvalues are very complicated and so have not been included here.

We need to find conditions that make the eigenvalues negative and the equilibrium point positive in order to determine the conditions that will make this equilibrium stable. In this case these conditions will tell us how our parameters should be related in order to ensure that the population does not contain any Fanatical Republicans but will instead the total population will be divided among the remaining groups. Unfortunately the way our parameters are related is too complicated to allow us to draw conclusions at this time. We instead perform a sensitivity analysis using this equilibrium point in the next section.

5.3 Sensitivity Analysis

The stability of E_6, cannot be easily determined and is very difficult to analyze, both numerically and analytically. This, coupled with the lack of quantitative data, makes describing the equilibrium point almost impossible. However, in many modeling situations where quantitative data is not readily available, and an equilibrium point is the only source of information that we have about a model, sensitivity analysis can give us some much needed information about the importance of all parameters in the system of equations. Sensitivity analysis allows us to determine which parameters will have the largest impact on the entire system. This information will help us determine which interactions have the most impact to our specified case.

The sensitivity (denoted Υ)of a dependent variable, y, with respect to an independent variable, x, is defined as:

$$\Upsilon_x^y = \frac{\partial y}{\partial x}\frac{x}{y}$$

This formula can be easily calculated given an explicit expression of y written in terms of x. However, if there isn't an explicit expression for the independent variables that are in terms of the parameters (as in our case), we must use an alternative way of calculating the sensitivity of an independent variable.

We consider the equilibrium point, E_6:

$$\left(\left(\frac{p_4}{b_{14}}, \frac{p_9}{c_{12}}, \frac{-p_4b_3p_9b_{14} + p_5p_9b_{14}^2 - p_4c_{12}b_4b_{14} + b_4c_{12}p_4^2 + p_9p_4b_4b_{14}}{b_{14}c_{12}b_4p_4}, \frac{p_9(-p_4b_3 + p_5b_{14})}{p_4c_{12}b_4}\right)\right($$

Since we don't have information about the analytical stability of this equilibrium point and only have conditions for its stability for specific parameters, it would be appropriate to instead calculate the most relevant parameters in the model for this equilibrium point.
Given a system of equations $[f_1, f_2, .., f_n]$ each defined by a set of variables, $[y_i]$, and a set of parameters, $[x_i]$, of the form $[y_1, y_2, .., y_n, x_1, x_2, .., x_n]$ and a specific parameter,p, alternative calculation then can be defined as:

$$\underline{\Upsilon^{y_x}} = J^{-1}\underline{c}, \frac{x}{\underline{y}},$$

where J is the Jacobian matrix of the system evaluated at the equilibrium point, c is a vector whose c_i entry corresponds to $\frac{\partial f_i}{\partial p}$ and Υ_x^y is the sensitivity vector, written in the form $[dS, dE_D, dE_R, dF_D, dF_R]$ Υ_x^y where the i_{th} entry of the vector is the sensitivity of the associated position. For example, if the sensitivity vector were $[1, 2, 3, 4, 5]^T$ then the sensitivity of the variable E_R with respect to the given parameter would be 3.
The problem arises when attempting to solve for the sensitivity vector, $\underline{\text{x}}$. If J is not invertible, then we cannot find a unique solution to the vector $\underline{\text{x}}$. Recall that there is a redundant equation in the system which will prohibit us from satisfying the requirement of the matrix being nonsingular. Thus, one of these equations, must be removed.
Let $\frac{dF_R}{dt}$ be the equation we remove from the system. To do this while preserving the system, F_r must be removed, as well. We can substitue F_R for $1 - S - E_D - E_R - F_D$, as follows from our closed population assumption. These two simplifications lead to a reduced 4×4 Jacobian matrix. More importantly, the removal of this redundancy allows the Jacobian matrix to be invertible.
The sensitivity vector results are presented in table 5.3.

Parameters	Corresponding r-vector
b_3	$\left[0, 0, \frac{p_9(-p_5 b_{14} + p_4 b_3)}{p_4 b_4^2 c_{12}}, \frac{p_9 p_4}{b_4 c_{12}}\right]^T$
b_4	$\left[0, 0, \frac{-p_9 p_4}{b_4 c_{12}}, \frac{p_9 p_4}{b_4 c_{12}}\right]^T$
b_{14}	$\left[0, 0, \frac{-p_9 p_4}{b_4 c_{12}}, \frac{p_9 p_4}{b_4 c_{12}}\right]^T$
b_{15}	$[0, 0, 0, 0]^T$
p_5	$\left[0, 0, \frac{b_{14} p_9}{p_4 b_4 c_{12}}, \frac{-b_{14} p_9}{p_4 b_4 c_{12}}\right]^T$
p_4	$\left[\frac{-1}{b_{14}}, 0, \frac{p_4^2 c_{12} b_4 - p_5 p_9 b_{14}^2}{b_{14} c_{12} b_4 p_4^2}, \frac{p_5 p_9 b_{14}}{b_4 c_{12} p_4^2}\right]^T$
p_9	$\left[0, \frac{1}{c_{12}}, -\frac{-p_4 b_3 + p_5 b_{14} + p_4 b_4}{b_4 p_4 c_{12}}, -\frac{p_4 b_3 + p_5 b_{14}}{b_4 p_4 c_{12}}\right]^T$
c_{12}	$\left[0, -\frac{p_9}{c_{12}^2}, -\frac{p_9(-p_4 b_3 + p_5 b_{14} + p_4 b_4)}{b_4 p_4 c_{12}^2}, \frac{p_9(-p_4 b_3 + p_5 b_{14})}{b_4 p_4 c_{12}^2}\right]^T$
p_{13}	$[0, 0, 0, 0]^T$
d_1	$[0, 0, 0, 0]^T$

The analysis reveals some interesting facts about the model. First, the vectors corresponding to the parameters b_{15}, p_{13}, and d_1 all yield the zero vector. This implies that each dependent

variable is completely insensitive to the parameters b_{15}, p_{13}, and d_1. Therefore, any value can be assigned to these parameters without fear that it will cause a major impact on the model both componentwise, or as a whole. This shows that the interactions corresponding to these terms have absolutely no impact on the equilibrium point, and more importantly, they do not change the possibility of the existence of a stable solution.
Secondly, the most important parameters to the equilibrium point E_6 are p_4, p_9 and c_{12}. This means that these interaction terms correspond to the most significant interactions in the model. Changes in any of these parameters will lead to the biggest impact to the variables that correspond to nonzero values in their respective vectors.
Many other dependent variables have sensitivity 1 with respect to varying parameters. This can be seen by identifying the position of any vector where these values occur. We can conclude that increases in these interactions occur in these parameters lead to the same magnitude of change in the dependent variable.
With this information, we can determine which parameters require the highest accuracy to the equilibrium point and which parameters can be picked with no attention to accuracy.

6 The Influence of Personal Influence Parameters

As we continue to qualitatively analyze our model, we now consider the effect of personal influence parameters. Personal influence can cause forward movement (as a conviction or self-motivation parameter) or backward movement (as an uncertainty parameter). For example, a personal influence parameter that causes forward movement is p_3; this parameter represents the rate at which individuals move from the Susceptible group to the Moderate Republican Group due to their personal beliefs. Similarly, p_4 is an uncertainty parameter; this parameter represents the rate at which individuals leave the Moderate Republican Class to join the Susceptible Group due to their personal feelings and not due to interaction with others. The cases that follow demonstrate the effect on our system of removing these parameters.

6.1 Absence of Personal Conviction Parameters

In this example, we do not consider personal influence as a cause for forward movement. Specifically, we set $p_3 = p_6 = p_{10} = p_{12} = 0$. By reducing the personal parameters, we now consider the IFE or idea-free equilibrium, as defined earlier.

Since the Susceptible group is under examination, we can now determine stability of the IFE by using the Jacobian matrix. The eigenvalues are:

$$\lambda_1 = 0,\ \lambda_2 = -p_{13},\ \text{and}\ \lambda_3 = -p_9.$$

λ_4 and λ_5 are very complicated. However, from them we find that the IFE is stable if:

$$b_{14} + b_3 > p_4 + p_8 + p_5 + p_7 \tag{14}$$

and

$$p_5p_4 + b_3b_{14} + p_5p_7 + p_8p_4 > b_3p_7 + p_5p_{14} + p_3p_4 + b_8b_1 + b_{16}p_7 + p_8b_{14}. \tag{15}$$

Since condition (14) shows $p_4 + p_5 > b_3 + b_{14}$, we have:

$$(b_3 - p_4) + (b_{14} - p_5) < 0.$$

Also, condition (15) can be rewritten for better understanding, thus

$$1 < \frac{(b_3 - p_5)}{b_1}\frac{(b_{14} - p_4)}{b_{16}}.$$

Interpretation of Stability Conditions

As usual, these stability conditions indicate which influences and interactions most significantly influence whether individuals tend to join or leave the Democratic and Republican parties. If the goal is to have individuals tend to join these two parties then political parties need to ensure that these stability conditions are not met. Once again, we note that in the two month period before an election it is unlikely that political parties can change the value of the personal influence parameters and focus our analysis on how political parties should attempt to manipulate the interaction parameters. Specifically, the first condition (14) indicates that once again the Republican and Democratic parties should ensure that their Moderate members recruit individuals at a higher rate than members leave due to personal influence. As we have come to expect, under these stability conditions, we find the surprising element that the Fanatical Influence is not significant in preventing a population from tending to avoid the Republican and Democratic Parties.

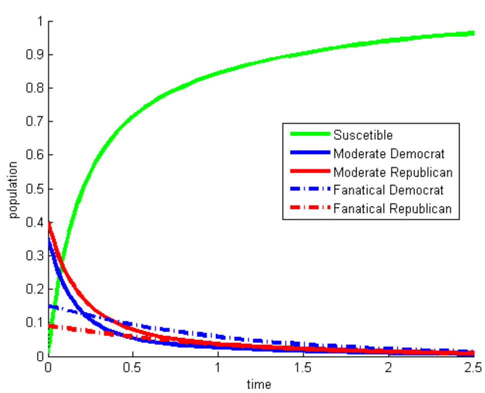


Figure 9: $p_3 = 4.2; p_4 = 1.6; p_5 = 2; p_6 = .2; p_7 = 1.4; p_8 = 1.9; p_9 = .1; p_{10} = .5; p_{12} = .7; p_{13} = 1.8$

In figure (9), a computational representation of the two given restrictions are demonstrated. Note that Figure (9) has limited conditions, but the classes and other parameters can be any value.Figure (9), illustrates that under these conditions both Moderate and Fanatical Groups will lose political interest.

6.2 Absence of Personal Uncertainty Parameters

Now, we are investigating the absence of personal uncertainty parameters. This means we do not consider personal influence as a cause for backward movement. Specifically, we consider $p_4 = p_5 = p_9 = p_{10} = 0$. By setting these personal influence parameters to zero we are eliminating all personal influence parameters that cause backward movement from the original system of differential equations.

As part of the analysis for this system, we find an IFE $(1, 0, 0, 0, 0)$. By evaluating the Jacobian at the IFE, we discover that one of the eigenvalues is 0 (as expected), but the other four are the roots of a fourth degree polynomial. (The coefficients of the polynomial consist of parameters.) Because of the difficulty of analytically finding the roots, we find these four other eigenvalues numerically. By plugging small numbers into the parameters, we always have one positive eigenvalue. Thus, we can determine that the this equilibrium point is a saddle, and thus unstable.

Interpretation Stability Conditions

Since this equilibria is unstable, then where there are no uncertainty personal influence parameters, there is no condition that will cause the population to tend to consist entirely of Susceptible Individuals. Thus, for this type of population (the only thing that causes an individual to become less active in politics is interaction with others), no matter how convincing the members of the Democratic and Republican parties are, as long as there are members of these parties to interact with others they will be able to convince a portion of the population to share their ideas. These results confirm the belief that interaction and connecting with individuals directly is very important during a presidential campaign.

6.3 Absence of Conviction and Uncertainty Parameters

We further examine personal influence by removing all the personal influence parameters p_i, i.e, there are no conviction or uncertainty parameters. In this case, we assume that personal influence parameters will not effect the presidential election. Mathematically this means that $p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = p_9 = p_{10} = p_{11} = p_{12} = p_{13}$. Recall that p_3, p_6, p_7, and p_8 are personal influence parameters that represent the way in which people's background or preconceived notions affect their political affiliations, while p_{10} and p_{12} represent the rate at which individuals transition from the Moderate Class to the Fanatical Class, i.e., become more committed to their political party due to the passage of time. Further recall that p_9 and p_{13} represent the way in which individuals become less active in their party (transition from the Fanatical Class to the Moderate Class); this transition could be due to a change in an individual's personal life that leaves him or her less time to devote to the party. Finally remember that p_4 and p_5 represent the rate at which individuals leave the Moderate Class to become Susceptible Individuals; this change in affiliation could be due to a lack of time to vote or simply a lack of personal conviction.

We evaluate this system of differential equations to find that it contains an IFE, $(1, 0, 0, 0, 0)$. We use the Jacobian evaluated at the IFE to find our eigenvalues. They are:

$$\lambda_{1,2,3} = 0 \text{ and } \lambda_{4,5} = \tfrac{1}{2}(b_3 + b_{14} \pm \sqrt{b_3^2 - 2b_3b_{14} + b_{14}^2 + 4b_{16}b_1}).$$

In this case we find that we will always have a positive eigenvalue. Hence, we have determined that the equilibrium is a saddle, which is unstable.

Interpretation of Stability Conditions

This means that the population of S will drop and level out at a particular value and since S

decreases, the other classes increase to some positive percent of the population and will all level off to some respective values. This is true because the system does not have a birth or death rate.

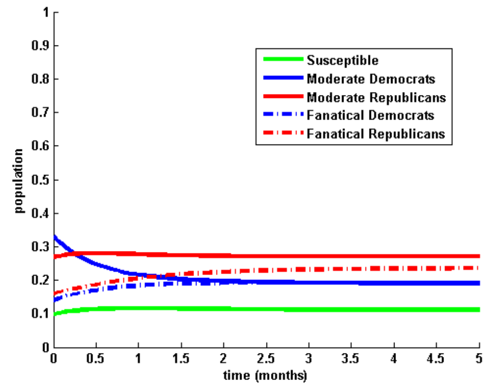


Figure 10: This is a numerical representation of a system in which there is no personal influence

Hence, our model shows that if interactions are exclusively responsible for all political affiliations, then we will again always have individuals affiliated with the Democratic and Republican parties provided that the population initially included a few Democrats and Republicans. These results, like the results from the last section, confirm the belief that interaction and connecting with individuals directly is very important during a presidential campaign.

6.4 Only Personal Influence Parameters

We continue to examine the effect of personal influence with one more example. This time we remove all the other types of influence and consider only personal influence parameters (both conviction and uncertainty personal influence parameters) which are denoted by p_i. In this case all other parameters are set equal to zero. This means there is no interaction between individuals in the system that can cause an individual to change their mind or political affiliation. The system of differential equations that results from this assumption is as follows.

$$\frac{dS}{dt} = -p_6 S - p_3 S + p_5 E_D + p_4 E_R \quad (16)$$

$$\frac{dE_D}{dt} = p_6 S - p_5 E_D + p_7 E_R - p_{10} E_D - p_8 E_D + p_9 F_D \quad (17)$$

$$\frac{dE_R}{dt} = p_3 S - p_4 E_R - p_{12} E_R + p_8 E_D - p_7 E_R + p_{13} F_R \quad (18)$$

$$\frac{dF_D}{dt} = p_{10} E_D - p_9 F_D \quad (19)$$

$$\frac{dF_R}{dt} = p_{12} E_R - p_{13} F_R \quad (20)$$

From this system of equations we have one equilibrium point:

$$\left(W, W\frac{(p_7p_3+p_4p_6+p_7p_6)}{p_4p_5+p_4p_8+p_7p_5}, W\frac{(p_5p_3+p_8p_3+p_8p_6)}{p_4p_5+p_4p_8+p_7p_5}, W\frac{p_{10}(p_7p_3+p_4p_6+p_7p_6)}{p_9(p_4p_5+p_4p_8+p_7p_5)}, W\frac{p_{12}(p_5p_3+p_8p_3+p_8p_6)}{p_{13}(p_4p_5+p_4p_8+p_7p_5)} \right), \tag{21}$$

where $W = S^* = 1 - E_D^* - E_R^* - F_D^* - F_R^*$. This substitution is possible because the total population $N = 1 = S + E_D + E_R + F_D + F_R$ is constant. The IFE, $(1, 0, 0, 0, 0)$ is only an equilibrium point if $p_3 = p_6 = 0$, and so we are not considering the IFE at this time.

The eigenvalues of the endemic equilibrium point, (21), were computed using the Jacobian matrix and thus we numerically analyzed the stability by setting all personal parameters less than .1. We found that the eigenvalues were strictly negative no matter how much we altered their values. This seems to indicate that the endemic equilibrium point, (21), is a stable point. This means that when there is no interaction between individuals and people make all their own decisions based on their personal preferences then the population approaches the state in which everyone is distributed among the voting classes.

We further investigate this case by eliminating the redundant equation. Recall that our model has a redundancy in the system do to the no birth or deaths, i.e., our total population in constant. So, we will make $W = 1 - E_D - E_R - F_R - F_D$. By doing this we can reduce the five equation system to a four equation system. We get new equations:

$$\frac{dE_D}{dt} = p_6(1 - E_D - E_R - F_R - F_D) - p_5E_D + p_7E_R - p_{10}E_D - p_8E_D + p_9F_D \tag{22}$$

$$\frac{dE_R}{dt} = p_3(1 - E_D - E_R - F_R - F_D) - p_4E_R - p_{12}E_R + p_8E_D - p_7E_R + p_{13}F_R \tag{23}$$

$$\frac{dF_D}{dt} = p_{10}E_D - p_9F_D \tag{24}$$

$$\frac{dF_R}{dt} = p_{12}E_R - p_{13}F_R \tag{25}$$

With this new system, we solve the system of equations and find a parameter based endemic equilibrium that is very long (so we do not show it here). We use the Jacobian and find out that the Jacobian does not have variables within it, thus evaluating the endemic equilibrium will not change the result. We find the eigenvalues on the general Jacobian and find that the eigenvalues are the roots of a fourth polynomial. We look for a zero-eigenvalue bifurcation by substituting $\lambda = 0$ in the characteristic equation. We find that there was no switch in sign but instead always remained negative. Thus we numerically concluded that we have a stable endemic equilibrium point.

Interpretation of Stability Conditions

So, we further believe the unstable equilibrium indicates that if all movement between classes is due to personal influence parameters, everyone will be distributed among the voting classes. This means that when there is no interaction between individuals and people make all their own decisions based on their personal preferences then the population approaches the state in which one party or group does not dominate the system.

7 Conclusions

We have analyzed five general cases and many subcases, which simplify our general mathematical model in this paper. In each example, we only consider some of the possible interactions and influences, mathematically represented by certain parameters set equal to zero. As we have shown, these different cases result in different equilibria and stability conditions. Since the results in each case vary significantly, we conclude that the absence and presence of different influences and interactions govern political affiliation over time. However, in the cases where the IFE exists, we have similar stability conditions. In order to give an idea of the significance of these different types of influences we review the results of each of the cases we analyzed.

In Section 3, the symmetric case, we were able to create a mathematical representation of a population similar to the American voting population before the 2000 presidential election. Specifically, we developed a model that represents a population that is almost evenly divided, with half the population supporting one party and half supporting the other. After analyzing this case, we found that depending on which interactions and influences most strongly affect individuals, the overall population will either lose all interest in the Democrat and Republican Parties, evenly divide themselves between only the Democratic Party and the Republican Party, or spread themselves out among all the groups, but with equal numbers of Democrats and Republicans. The results of this section were not particularly surprising; however, it is interesting to note that Moderate Individuals in both parties have a strong influence on the political affiliations formed. This suggests that in a dual party system, political activists might want to focus resources on interacting with individuals in a less fanatical or active manner. Instead, they might want to focus on creating situations in which undecided individuals are able to ask questions about politics and have their questions answered in a moderate manner. (We have assumed that moderates will discuss politics if asked but do not instigate political debates.)

Recall that in Section 4, we considered individuals who can only become more politically active and can only switch parties due to personal beliefs. We found that the whole population eventually becomes Fanatical. As such, this shows that over time personal influence is not enough to keep membership in the Moderate groups. We note that the speed with which the population becomes entirely Fanatical depends on the non-zero parameters. Hence, if there were populations that always increase political activity, then political activists would only need to insure there were convincing interactions between members of the political party and other individuals. Insuring this corresponds to an increase of the relevant parameter values and causes the entire population to become active more quickly, i.e., within the two-month period of the presidential campaign.

We also considered individuals who decrease political involvement only due to personal beliefs and may not directly switch affiliations in Section 5. This time we found that the ultimate distribution of the population between voting classes depends on how persuasive the Moderates are. Specifically we found that the overall population will either: lose all interest in the Democratic and Republican Parties, be divided between the Susceptible Group and either Moderate Democrats or Moderate Republicans, consist of all classes except one of the Fanatic Classes, or be distributed among all the Moderate Classes. From our analysis thus far, we learn that in a population of this type, the moderate class has the most influence on the formation of political affiliations. Thus, we conclude that in this situation political activists should focus their resources on the Moderate Classes. For example, the more fanatical members might want to be less pushy or flashy in their efforts to convince others to join a particular party and instead act as Moderates do and allow others to approach them with questions about politics. Efforts could be made to promote low-pressure exchanges of political ideas, in which content is more important than loudly spreading political rhetoric.

In the latter parts of the paper we investigated the effect personal influence has on the formation of political affiliations (Sect 6). We assume that political parties cannot a ect how a person's personal beliefs influence them during the two-month campaigning period we consider. However, investigating the effects of these parameters gives us insight into which types of interactions most influence people to form political affiliations when they are also influenced by particular types of personal beliefs.

Specifically, we first considered the absence of personal conviction parameters (Section 6). In this case we found that if the Moderates and Fanatics are not sufficiently convincing and engaging the overall population will eventually lose all interest in the Democrat and Republican Parties. We also see that negative interactions and interactions with Moderates seem to have a greater influence on convincing people to join the two main parties. So, in order to draw people toward the Democratic and Republican Parties, once again political activists may want to focus on increasing the effectiveness of interactions, especially those with Moderates.

We also investigated what happens when personal uncertainty parameters do not affect the formation of political affiliations (Section 6). We found that if the only thing that causes an individual to become less active in politics is interaction with others then the population will never consist of only Susceptible Individuals. This means that for this type of population, no matter how convincing the members of the Democratic and Republican parties are, as long as there are members of these parties to interact with others they will be able to convince a portion of the population to share their ideas. Additionally, our model shows that if interactions are exclusively responsible for all political affiliations, then we will again always have individuals affiliated with the Democratic and Republican parties provided that the population initially included a few Democrats and Republicans.

We further discovered, in Section 6, that if all movement between classes is due to personal influence parameters, everyone will be distributed among the voting classes. This means that when there is no interaction between individuals and people make all their own decisions based on their personal preferences then the population approaches the state in which one party or group does not dominate the system.

Our final analysis in Section 7 investigated what happens if all influences and interactions affect people in the same way. We see that if all influences and interactions are identical, then the overall population tends to almost evenly distribute itself in all the voting classes. Hence, to fully use our research a political analyst must first understand what motivates the people they wish to influence. Once they understand the impacting influences and interactions they may choose a case that most closely reflects their situation. The analysis of the case will help indicate to the political analyst which influences and interactions are the most important in recruiting people. Additionally, we note that in many cases the Moderates had the most effect on the political affiliations formed during our time period. This suggests that the political parties may be better served by devoting fewer resources to fanatical campaigning. Instead, our model suggests that creating forums for calm and well supported discussion (i.e. Moderate interaction) may be a more effective form of recruitment.

8 Future Work

After reviewing the respective equilibrium values in each case in the model, we see that it may be beneficial to focus on certain parameters and evaluate the relevance of each one. So we could conduct additional sensitivity analysis with respect to the independent variables that each parameter a ects in the future. It would also be important to search for any other trends in each of the cases and use this information to come to additional logical conclusion about the model in general. For example, our analysis indicates that in many cases, there are certain conditions that will cause the population to lose all interest in the Democrat and Republican Parties (Shown when the equilibrium point $(1, 0, 0, 0, 0)$ is stable). From, this we conclude that populations often naturally tend toward disinterest in the major political parties. Furthermore, connecting with people is very important if a Democratic or Republican presidential candidate wants to win an election or a political analyst wishes to increase voter turn-out. However, we would also like to note that time is a factor in this case. For example, under many parameter regimes it would take years to reach the state where there are only Susceptible Individuals in the population. However, if one could find realistic parameter values that cause the population to consist of only Susceptible Individuals in less than two-months (the time we allow for campaigning before a presidential election), then it would be possible for a third party candidate to win a US Presidential Election. This has not happened in over fifty years; so, either it is impossible to have parameters values of this type in the modern world, or it is very unlikely. This is an avenue that we would like to research further in the future. Also, with some cases, a graphical representation would be best to fully understand the stability at a given equilibrium point. We hope to find numerical parameter values to validate the legitimacy of our model. Thus, our biggest goal for future work is to draw back to the original application of the model to have a clearer understanding about what our mathematical calculations show about the voting system.

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10 Appendix 1:

Here's a summary of some of the relevant equilibria and their stability for various cases.

10.1 Summary of Results in Section 3

Case	Equilibrium Points	Stability Conditions
Symmetric Case (3.1.1) $(1, 0, 0)$ with p_i=0 $\forall i$	$(1, 0, 0)$	always unstable
Symmetric Case (3.1.2-3.1.3)	$(1, 0, 0)$ where $p_3 = 0$	$p_3 = 0$ $\frac{p_9}{p_9 p_4 - p_{10}} < 1$
	$(0, 1/2, 0)$ where $b_7 = g_1 = p_4 = p_{10} = 0$	$b_7 = g_1 = p_4 = p_{10} = 0$ $\frac{g_2 + c_{12}}{c_9 + 2p_9} < 1$
	$\left(\left(0, 0, \frac{1}{2}\right)\right($ where $c_{10} = p_9 = 0$	$c_{10} = p_9 = 0$ $\frac{c_9}{c_{12} + g_2 + 2p_{10}} < 1$
	$f(V, A, g_1, \quad , \quad , p_3, g_2, \nu, \rho, c_{10}, p_9, \quad , \quad 1, \epsilon, p_{10})$ without parameter restrictions	This case is stable under certain parameter values (numeric analysis)

10.2 Summary of Results in Section 4

Case	Equilibrium points	Stability Conditions
Strong Political Interest and Assorted Subcases	$(0, 0, 0, x, 1 - x)$	always center-stable

10.3 Summary of Results in Section 5

Case	Equilibrium points	Stability Conditions
5	$\frac{N^- + N^+}{D^- + D^+}, B, Z, \frac{p_{10}B}{Bc_{12} - p_9}, \frac{p_{12}Z}{-d_1Z + p_{13}}\Big)\Big($	stable for certain parameters (numerical analysis)
5.1	$(1, 0, 0, 0, 0)$	$\frac{b_{14} + b_3}{p_5 + p_7 + p_8 + p_4} < 1$ $\frac{p_7}{b_{14} - p_4} + \frac{p_8}{b_3 - p_5} < 1$
5.2.1 (E_1)	$(1, 0, 0, 0, 0)$	$p_5 > b_3$ and $p_4 > b_{14}$
5.2.2 (E_2)	$\Big(\Big(\frac{p_4}{b_{14}}, 0, 1 - \frac{p_4}{b_{14}}, 0, 0\Big)\Big($	$p_4 < b_{14}$, $p_4b_3 < p_5b_3$, and $d_1b_{14} < d_1p_4 + b_{14}p_{13}$; $b_{14} \geq p_4$ for existence
5.2.3 (E_3)	$\Big(\Big(\frac{p_5}{b_3}, 1 - \frac{p_5}{b_3}, 0, 0, 0\Big)\Big($	$p_5 < b_3$, $p_5b_{14} < p_4b_3$, and $c_{12}b_3 < c_{12}p_5 + b_3p_9$; $b_3 \geq p_5$ for existence
5.2.4 (E_6)	$\Big(\Big(\frac{p_4}{b_{14}}, \frac{p_9}{c_{12}}, Z, \frac{-p_9(p_4b_3 - p_5b_{14})}{(p_4b_4c_{12})}, 0\Big)$	$\Big($We need to find conditions that make the eigenvalues negative and the equilibrium relevant. We will perform a sensitivity analysis to aid us.

10.4 Summary of Results in Section 6

Case	Equilibrium points	Stability Conditions
Absence of Personal Conviction (6.1)	$(1, 0, 0, 0, 0)$	$(b_3 - p_4) + (b_{14} - p_5) < 0$ $1 < \frac{(b_3 - p_5)}{b_1} \frac{(b_{14} - p_4)}{b_{16}}$
Absence of Personal Uncertainty (6.2)	$(1, 0, 0, 0, 0)$	unstable
Absence of Personal Conviction and Uncertainty (6.3)	$(1, 0, 0, 0, 0)$	unstable
Only Personal Influence (6.4)	$W, \frac{W(p_7p_3+p_4p_6+p_7p_6)}{p_4p_5+p_4p_8+p_7p_5}, \frac{W(p_5p_3+p_8p_3+p_8p_6)}{p_4p_5+p_4p_8+p_7p_5}, \frac{Wp_{10}(p_7p_3+p_4p_6+p_7p_6)}{p_9(p_4p_5+p_4p_8+p_7p_5)}, \frac{Wp_{12}(p_5p_3+p_8p_3+p_8p_6)}{p_{13}(p_4p_5+p_4p_8+p_7p_5)}\Bigg)\Bigg($	stable according to numerical investigation

More information about the all tables can be found in the sections referenced in the tables.

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